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Semiclassical relativistic strings in S^5 and long coherent operators in $\mathcal{N}$=4 SYM theory

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ABSTRACT: We consider the low energy effective action corresponding to the 1-loop, planar, dilatation operator in the scalar sector of $\mathcal{N} = 4$ $SU(N)$ SYM theory. For a general class of non-holomorphic "long" operators, of bare dimension $L \gg 1$, it is a sigma model action with 8-dimensional target space and agrees with a limit of the phase-space string sigma model action describing generic fast-moving strings in the S^5 part of $AdS_5 \times S^5$. The limit of the string action is taken in a way that allows for a systematic expansion to higher orders in the effective coupling $\tilde{\lambda} = \frac{\lambda}{L^2}$. This extends previous work on rigid rotating strings in S^5 (dual to operators in the $SU(3)$ sector of the dilatation operator) to the case when string oscillations or pulsations in S^5 are allowed. We establish a map between the profile of the leading order string solution and the structure of the corresponding coherent, "locally BPS", SYM scalar operator. As an application, we explicitly determine the form of the non-holomorphic operators dual to the pulsating strings. Using action–angle variables, we also directly compute the energy of pulsating solutions, simplifying previous treatments.

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Contents

1. Introduction

The AdS/CFT correspondence [1] gave a precise example of the conjectured relation [2] between the large N limit of gauge theories and string theory. In its most well-known form it claims that $\mathcal{N}=4$ SYM theory with gauge group $SU(N)$ and coupling $g_{\rm YM}$ on one hand, and type IIB string theory on $AdS_5 \times S^5$ with N units of RR 5-form flux, and string coupling $g_s = g^2_{\rm YM}$ on the other, are just two different descriptions of the same theory. The string theory becomes weakly coupled, i.e. the theory becomes "stringy", in the limit $g_{\rm YM} \to 0$, $N \to \infty$, with the 't Hooft coupling $\lambda = g^2_{\rm YM} N$ being fixed and large, $\lambda \gg 1$.

It remained unclear, however, how strings "emerge" from the field theory, in particular, which (local, single-trace) gauge theory operators [3] should correspond to which "excited" string states and how one may verify the matching of their dimensions/energies beyond the well-understood BPS/supergravity sector. An important step in that direction was made in [4] where it was shown how this correspondence can be established for a class of "small" (nearly point-like) near-BPS strings which are ultrarelativistic, i.e. whose kinetic energy is much larger than their mass.

Shortly after, in [5] it was argued that at least a qualitative agreement between the non-BPS states on the two sides of the duality can be established also for certain extended string states represented by classical string solutions with one large AdS_5 angular momentum. The semiclassical approach of [5] was further developed and generalized to multispin string states in [6, 7]. It was proposed in [7] that, like for "small" near-BPS BMN [4] string states, a *quantitative* agreement between string theory and gauge theory should be found also for a class of extended classical solutions with two or three non-zero angular momenta in the S^5 (see [8] for a review). For such solutions the classical energy has a regular expansion in powers of $\tilde\lambda = \frac{\lambda}{L^2}$, where $L = J$ is the total S^5 spin, $E = L(1 + c_1\tilde\lambda + c_2\tilde\lambda^2 + ..)$, and the string $\alpha' \sim \frac{1}{\sqrt{\lambda}}$ corrections are suppressed in the limit $L \to \infty$, $\tilde\lambda$ =fixed. Assuming that the large L and then small $\tilde\lambda$ limit is well-defined also on the SYM side, one should then be able to compare the classical string results for the energy to the quantum SYM results for the corresponding anomalous dimensions.

Using the crucial observation of [9] that the one-loop scalar dilatation operator can be interpreted as a Hamiltonian of an integrable $SO(6)$ spin chain and thus can be diagonalized even for large L by the Bethe ansatz method, the proposal of [7] was confirmed at the leading order of expansion in $\tilde\lambda$ in [10, 11]. There, a remarkable agreement was found between energies of various string solutions and eigenvalues of the dilatation operator representing dimensions of particular SYM operators.

The established correspondence was thus between a thermodynamic limit $L \to \infty$

of the Bethe ansatz eigenstates of the integrable spin chain and a large spin, or, equivalently, large energy, limit of the classical solitonic solutions of the string sigma model action. The classical bosonic coset sigma model also has a well-known integrable structure which becomes explicit for particular rigid-shape rotating string configurations [12, 13] and which can be mapped [14, 15] to the one of the spin chain.

The next important step was made in [16] where it was shown that one can take the large energy, or ultrarelativistic, limit directly in the string action getting a reduced "non-relativistic" sigma model that describes in a universal way the leading-order $O(\tilde{\lambda})$ corrections to the energies of all string solutions in the two-spin sector. Moreover, it was found [16] that the resulting action agrees exactly with the semiclassical coherent state action describing the $SU(2)$ (Heisenberg $\mathrm{XXX}_{1/2}$) sector of the spin chain in the $L \to \infty$, $\tilde{\lambda}$ =fixed limit, which turns out to be equivalent to a continuum limit in which one keeps only linear and quadratic derivative terms. This demonstrated how a string action can directly emerge from a gauge theory in the large-N limit and provided a direct map between "coherent" SYM states or operators built out of two holomorphic scalars, and all two-spin classical string states, bypassing the need to apply the Bethe ansatz to find anomalous dimension for each particular state. Furthermore, the correspondence established at the level of the action implies also the matching of fluctuations around particular solutions (just as in the BMN case where one matches fluctuations near a BPS state) and thus goes beyond the case of rigidly rotating strings. This remark applies also to other subsectors which we shall describe below which therefore overlap, i.e. are related by small deformations. These subsectors are named according to a certain type of basic solutions they contain (which carry several large conserved charges) but they also describe many "nearby" states which may be labeled by higher conserved charges or oscillation numbers.

The observation made in [16] may be viewed also as based on the fact that the spin chain has two equivalent descriptions, a Hamiltonian operator one and a Lagrangian path integral one. The corresponding Lagrangian was shown to be identical to a limit of the classical string Lagrangian. The relevant semiclassical limit of the path integral is, as usual, naturally represented by the coherent states of the operator approach. The matching demonstrated in [10, 11] can then be interpreted as an equivalence between the two descriptions of the spin chain. The remarkable fact is that the classical solutions in the path integral approach, namely the solutions of the Landau-Lifshitz (LL) equations, are in correspondence with exact eigenstates found using the thermodynamic limit of the Bethe ansatz: the energies as well as all higher conserved charges are the same. The general proof of this fact was given later in [17] using integrable models methods.

From the effective action point of view, one can also argue that, to lowest order in derivatives, there is only one unknown coefficient that can be fixed, e.g., by comparison

with the BMN [4] result. Therefore, at leading order in the $L \to \infty$, $\tilde{\lambda}$ =fixed limit, the effective action is unique and should be expected to reproduce the same limit of the exact results. This uniqueness is lost at higher orders in $\tilde{\lambda}$ where more terms in the effective action are present. In that case, the coherent state approach needs to be generalized as was explained in [18]. This allowed us to verify the correspondence at the $\tilde{\lambda}^2$ order. An equivalent general result (using the integrable spin chain embedding of [19]) was obtained also in the Bethe ansatz approach [17].

The approach of [16, 18] was also generalized (to leading order in $\tilde{\lambda}$) to the three-spin or $SU(3)$ sector [20, 21] (as well as to the $SL(2)$ [22, 11] sector corresponding to one spin in AdS_5 and one in the S^5 [21]). On the Bethe ansatz side the agreement between the energies of particular 3-spin string solutions and the corresponding spin chain eigenvalues was previously shown in [15, 23].

The results reviewed above explained the matching between all S^5 rotating string solutions with 3 large angular momenta and all "long" operators constructed out of the three scalars $X = X_1 + iX_2$, $Y = X_3 + iX_4$, $Z = X_5 + iX_6$ (including, as mentioned above, also "near-by" states). However, there is also another interesting class of solutions for strings moving in S^5 – the so called pulsating strings [24, 15] which also have a regular expansion of their energy in terms $\tilde{\lambda} = \frac{\lambda}{L^2}$ where L is a large "oscillating number". Their energies were matched (to order $\tilde{\lambda}$) to the energies of the SYM theory states in [10, 15] using the Bethe ansatz techniques for the $SO(6)$ spin chain of [9].

To carry out a similar matching at the level of the effective action, that is to match the corresponding coherent states and not only the energy eigenvalues,[1] we need to understand how to extend the ideas discussed above in the rotating string $SU(3)$ sector to the whole $SO(6)$ spin chain [9], i.e. to the subset of the SYM operators constructed out of all 6 real scalars and not limited to the holomorphic products of X, Y, Z.

The first step towards that goal was made in [21]. There, the Grassmanian $G_{2,6} = SO(6)/[SO(4) \times SO(2)]$ was identified as the coherent state target space for the spin chain sigma model since it parametrizes the orbits of the half-BPS operator $X = X_1 + iX_2$ under the $SO(6)$ rotations. In a related development, motivated by the suggestions in [25] and [16], the procedure of taking the high energy or $\tilde{\lambda} \to 0$ limit of the classical string theory was generalized [26, 27] to the whole $AdS_5 \times S^5$ bosonic action (and was later extended to include fermions [28]).

In the present paper we shall study the $SO(6)$ sector in detail, carefully working out the spin chain and the string theory sides of the correspondence. We will show

[1] The Bethe ansatz approach [9, 10, 15] provides, in principle, a recipe to construct the corresponding pure eigenstates or Bethe wave functions, but this is not easy to do in practice.

that the agreement between the two effective actions extends to the whole subsector of scalar operators characterized by a "local BPS" condition, i.e. built out of products of $SO(6)$ rotations of the BPS 6-vector $(1, i, 0, 0, 0, 0)$. It is this condition that selects the coset $G_{2,6}$ as the target space. This condition ensures that the corresponding anomalous dimensions on the field theory side are of order $\frac{\lambda}{L}$ and thus can be compared to the leading order corrections to the energy on the string side. The role of this locally BPS condition was also emphasized in [27].

On the string side, we need to find a "reduced" sigma model by taking a large energy limit of the classical string action. We shall essentially follow [16, 18, 21] but improve the derivation of the reduced action in two ways. First, we will clarify the gauge fixing procedure by using an alternative, 2-d dual (or "T-dual") action where the linear in time derivative "Wess-Zumino" term appears more naturally from the usual B_{mn}–field coupling term. Second, we will use canonical transformations to systematize the change of variables that was previously needed [18] to eliminate terms of higher than first power of time derivatives. In this way will we find a completely systematic and universal procedure to derive higher order in $\tilde{\lambda}$ corrections on the string theory side. The procedure is independent of a particular solution one may consider and, moreover, it should be possible to generalize this procedure to the full $AdS_5 \times S^5$ case.

We should mention that the method of canonical perturbations was already applied to this problem in ref. [27] which computed the leading order action for a generic fast motion in $AdS_5 \times S^5$ using a similar but somewhat different approach based on consideration of near light-like surfaces. The advantage of our procedure is in its systematic nature which allows us to compute higher order corrections with relative ease.

On the spin chain side, we will use a variation of the coherent state approach that was successful in the $SU(2)$ and $SU(3)$ case. In the coherent state approach, one first reformulates the quantum mechanical spin chain problem in terms of a coherent state path integral [29] and then observes that in the limit we are interested in, i.e. $L \to \infty$, $\tilde{\lambda}$ =fixed, one can take the continuum limit and all quantum corrections can be ignored. This is essentially equivalent to ignoring quantum mechanical correlations between different sites of the spin chain, and, as a result, we are lead to a classical action for the system. The variation of the coherent state approach we shall use is simply to ignore quantum correlations from the very beginning by considering states which are the product of independent states at each site. The classical action can then be thought of as the action that leads to the Heisenberg equations of motion in this restricted subspace. This method, which is equivalent to the one used in [18], leads to the same result at leading order but has some practical advantages when applied at the next order.

In this paper we will only consider the leading order term in the spin chain effective action which turns out to be in perfect agreement with the leading order action obtained on the string side.

An important consistency check is that, starting with the reduced sigma model, we should be able to reproduce the leading order results for all pulsating and rotating string solutions in $R_t \times S^5$. The map that emerges between the field theory and the string theory [16, 26, 27] is that each portion of the string that moves approximately along a maximum circle carrying one unit of R-charge, corresponds to a site of the spin chain at which there is a half-BPS operator with the same R-charge. In particular, this map allows us to find the coherent operators corresponding to the pulsating strings of [24, 15]. Since the pulsating solutions are a particular case, the agreement between the actions that we find here explains the agreement already observed in [10, 15] between the string result for the energy and the eigenvalues obtained using the Bethe ansatz for the $SO(6)$ chain. This also shows that in this case there is an exact agreement between the energy as a function of the conserved quantities as obtained from the solutions of our sigma model and the one obtained from the Bethe ansatz. This leads us to conjecture that one should be able to prove the agreement in general as was done in the $SU(2)$ case in [17].

As a side but interesting result, we also find, using action–angle variables, an exact classical relation between the energy and the constants of motion of pulsating solutions. This simplifies previous treatments, putting these solutions at the same level as the rotating ones of [7, 12, 13].

The organization of this paper is as follows. In section 2 we discuss the most general case of rotating strings in S^5 and the gauge fixing procedure. In the next section 3 we do the same for the most general fast motion on S^5 which includes also pulsating solutions. We also explain there a systematic procedure to compute higher orders in $\tilde{\lambda}$ from the string side. In section 4 we obtain the same leading order sigma model starting from the field theory side and studying the action of the dilatation operator on operators constructed out of scalars, i.e. starting from the $SO(6)$ spin chain Hamiltonian of [9]. We give examples of the string–spin chain correspondence in the following section 5 where we verify that it includes all known solutions in which the motion is on the S^5. Finally, in section 6 we give a detailed analysis of the pulsating solution of [15] using action–angle variables.

In section 7 we make some concluding remarks, and in the Appendix we give a brief introduction to the subject of canonical perturbation theory to make the paper self-contained and as a reference for the reader.

2. String theory side: rotating strings ($SU(3)$ sector)

In preparation for studying the full $SO(6)$ sector, which is done in the next section, let us start here by describing the general procedure to derive the effective action for a rotating string in the limit of large semiclassical rotation parameters. This connects and generalizes previous partial results of [16, 18, 20, 21]. We shall isolate and gauge-fix a "fast" collective coordinate and get an action for "slow" variables as an expansion in $\tilde{\lambda} = \frac{\lambda}{L^2}$, where $L = J$ is the total S^5 angular momentum. This action will thus reproduce the expression for the energy of rotating string solutions expanded in powers of $\tilde{\lambda}$ [7, 12, 13, 8]. In the two-spin or "$SU(2)$" sector it will coincide with the action found in [18], while in the most-general pure-rotation three-spin or "$SU(3)$" sector it will generalize the leading-order action found in [21, 20] (see also [27, 28]) to all orders in $\tilde{\lambda} = \frac{\lambda}{L^2}$. Here we shall explicitly consider only the case when string moves on S^5 but a generalization to the case when there is also a motion in AdS_5 is possible too (see [21] and references there).

2.1 Isolating the "fast" angular coordinate α

In the case of generic rotating strings it is natural to follow [7, 18, 21] and parametrize the S^5 metric in terms of 3 complex coordinates Z_i ($i = 1, 2, 3$)

$$ds^2 = -dt^2 + dZ_i^* dZ_i \,, \qquad\qquad Z_i^* Z_i = 1 \,, \tag{2.1}$$

where t is the time direction of AdS_5. In terms of 6 real coordinates X_i or standard angles of S^5 one has

$$Z_1 = X_1 + iX_2 = \sin\gamma \ \cos\psi \ e^{i\varphi_1} \ , \qquad Z_2 = X_3 + iX_4 = \sin\gamma \ \sin\psi \ e^{i\varphi_2} \ ,$$

$$Z_3 = X_5 + iX_6 = \cos\gamma \ e^{i\varphi_3} \ . \tag{2.2}$$

In this parametrization string states that carry three independent (Cartan) components J_i of $SO(6)$ angular momentum should be rotating in the three orthogonal planes [7, 8]. To consider the limit of large total spin $J = J_1 + J_2 + J_3$ we would like to isolate the corresponding collective coordinate, i.e. the common phase of Z_i. In the familiar case of fast motion of the center of mass the role of J is played by linear momentum or p^+. Here, however, J represents the sum of an "orbital" *and* "internal" angular momenta and thus does not correspond simply to the center of mass motion. This is thus a generalization of the limit considered in [4]: we are interested in "large" extended string configurations and not in nearly point-like strings.

Isolating the common phase in the three orthogonal planes by introducing the new coordinates α and U_i

$$Z_i = e^{i\alpha} U_i \, , \qquad\qquad U_i^* U_i = 1 \, , \tag{2.3}$$

one finds that the metric (2.1) becomes

$$ds^2 = -dt^2 + (d\alpha + C)^2 + dU_i^* dU_i - C^2 = -dt^2 + (D\alpha)^2 + DU_i^* DU_i \, , \tag{2.4}$$

where

$$C \equiv -iU_i^* dU_i \, , \quad D\alpha \equiv d\alpha + C \, , \quad DU_i \equiv dU_i - iCU_i \, , \quad DU_i^* \equiv dU_i^* + iCU_i^* . \tag{2.5}$$

Here U_i belongs to CP^2: the metric is invariant under a simultaneous shift of α and a rotation of U_i. In general, this parametrization corresponds to a Hopf $U(1)$ fibration of S^{2n+1} over CP^n: $ds^2 = DU_i^* DU_i$ is the Fubini-Study metric and $K = \frac{1}{2} dC$ is the covariantly constant Kähler form on CP^n. In the two-spin or $SU(2)$ sector ($Z_3 = 0$) where the motion is within S^3 of S^5 the two coordinates U_r of CP^1 can be replaced by a unit 3-vector [18]

$$n_i \equiv U^\dagger \sigma_i U \ , \qquad U = (U_1, U_2) \ , \qquad DU_r^* DU_r = \frac{1}{4} dn_i dn_i \ , \tag{2.6}$$

and C has a non-local WZ-type representation $C = -\frac{1}{2}\int_0^1 d\xi \ \epsilon_{ijk} n_i \partial_\xi n_j dn_k$.

The general form of the string action in $R_t \times S^5$ (the string is positioned at the center of AdS_5 with t being the AdS_5 time) with the metric (2.4) is then (we use world-sheet signature $(-+)$)

$$I = \sqrt{\lambda} \int d\tau \int_0^{2\pi} \frac{d\sigma}{2\pi} \ \mathcal{L} \ , \tag{2.7}$$

$$\mathcal{L} = -\frac{1}{2}\sqrt{-g} g^{pq} \big(- \partial_p t \partial_q t + D_p \alpha D_q \alpha + D_p U_i^* D_q U_i \big) \ . \tag{2.8}$$

The crucial point is that one should view t and α as "longitudinal" coordinates that reflect the redundancy of the reparametrization-invariant string description: they are not "seen" on the gauge theory side, and should be gauged away (or eliminated using the constraints). At the same time, the CP^2 vector U_i describing string profile should be interpreted as a "transverse" or physical coordinate which should thus have a counterpart on the spin chain side, with an obvious candidate being a vector parametrizing

the coherent state [16, 18, 21]. The conserved charges corresponding to translations in time and α are

$$E = \sqrt{\lambda}\mathcal{E} \ , \qquad \mathcal{E} = -\int_0^{2\pi} \frac{d\sigma}{2\pi} \sqrt{-g} g^{0p} \partial_p t \ , \tag{2.9}$$

$$J = \sqrt{\lambda}\mathcal{J} \ , \qquad \mathcal{J} = \int_0^{2\pi} \frac{d\sigma}{2\pi} \ P_\alpha \ , \qquad P_\alpha = -\sqrt{-g} g^{0p} D_p \alpha \ , \tag{2.10}$$

where the effective coupling constant $\tilde{\lambda}$ is directly related to the (rescaled) charge $\mathcal{J}$ in (2.10)

$$\tilde{\lambda} \equiv \frac{\lambda}{J^2} = \frac{1}{\mathcal{J}^2} \ , \quad \text{i.e.} \quad \mathcal{J} = \frac{1}{\sqrt{\tilde{\lambda}}} \ . \tag{2.11}$$

The "fast motion" expansion in powers of $\frac{1}{\mathcal{J}^2}$ which we will be interested in is, thus, the same as the expansion in powers of $\tilde{\lambda} \to 0$.

What remains is to do the following three steps: (i) proper gauge fixing; (ii) expansion of the action at large $\mathcal{J}$, suppressing time derivatives of U_i in favor of spatial derivatives; (iii) field redefinitions to eliminate from the expanded action terms of higher order than first in time derivatives.

To leading order in $\tilde{\lambda}$ one may simply use the standard conformal gauge (as was done in the $SU(2)$ sector in [16, 18] and in the $SU(3)$ sector in [21, 20]). As was pointed out in [18], to get the full action to all orders in $\tilde{\lambda}$ one should use a special "adapted" gauge. Here we shall make this first gauge fixing step particularly transparent by explaining that the gauge fixing procedure used in [18] amounts simply to the standard static gauge for the coordinate which is 2-d dual (or "T-dual") to α.

Having in mind comparison with the spin chain side it is natural to request that translations in time in the target space and on the world sheet should be related. Also, we should ensure that the angular momentum $\mathcal{J}$ is homogeneously distributed along the string so that its density P_α in (2.10), i.e. the momentum conjugate to α, is constant. Therefore, one should fix the following gauge [18]

$$t = \tau \ , \qquad P_\alpha = \mathcal{J} = \text{const} \ . \tag{2.12}$$

As was shown in [18], starting with the phase-space form of the string action and imposing this "non-conformal" gauge one finds the following effective Lagrangian for U_i

$$\mathcal{L} = \mathcal{J} C_0 - \sqrt{-\det \tilde{h}_{pq}} \ , \tag{2.13}$$

$$\tilde{h}_{pq} = \tilde{\eta}_{pq} + D_{(p}U_i^* D_{q)}U_i \ , \qquad \tilde{\eta}_{ab} \equiv \mathrm{diag}(-1, \mathcal{J}^2) \ , \qquad C_p = -iU_i^* \partial_p U_i \ . \quad (2.14)$$

In the $SU(2)$ case [18] this gives an equivalent action for n_i (2.6) with $\tilde{h}_{pq} = \tilde{\eta}_{pq} + \frac{1}{4}\partial_p n_i \partial_q n_i$. As was noted in [18], apart from the WZ-type term C_0, the Lagrangian (2.13) looks like a Nambu Lagrangian in a static gauge, suggesting that there may be a more direct way of deriving it. This is indeed the case as we shall explain below.

2.2 2-d duality transformation $\alpha \to \tilde{\alpha}$ and "static" gauge fixing

Let us make few remarks on interpretation of string action in connection with spin chain on the SYM side and for simplicity consider the $SU(2)$ case. In the semiclassical coherent state description of the spin chain [16], one has a circular direction along the chain at each point of which one has a classical spin vector $\vec{n}$ belonging to a 2-sphere. In other words, if we combine operator ordering direction under the trace with an "internal" direction we get $S^1 \times S^2$ type of geometry. A similar geometry indeed emerges on the string sigma model side – we have S^3 fibered by S^2 with base S^1. However, in (2.4) the S^1 direction is that of the angle α. Fast rotation corresponds to time-dependent $\alpha \sim t + \ldots$, so it is not quite appropriate to call α a "longitudinal" coordinate since after we have chosen $t = \tau$ gauge we have already fixed a time-like coordinate. A natural idea is that the true longitudinal coordinate should be "T-dual" counterpart $\tilde{\alpha}$ of α, i.e. one should apply 2-d duality to the scalar field α in (2.8).[2]

There is, however, an important subtlety. The standard discussions of T-duality are usually done in conformal gauge, but if we would fix the conformal gauge and then also $t = \tau$ we would no longer have a freedom to fix $\tilde{\alpha} \sim \sigma$. Actually, $\tilde{\alpha} \sim \sigma$ will not, in general, be a solution of the equations for $\tilde{\alpha}$ in the conformal gauge. The correct procedure is not to impose the conformal gauge; we should first apply the 2-d duality, then go from the Polyakov to the Nambu form of the action by solving for the 2-d metric g_{pq}, and finally fix the static gauge $t \sim \tau$, $\tilde{\alpha} \sim \sigma$. Remarkably, this turns out to be *equivalent* to the gauge fixing procedure used in [18], leading directly to (2.13). This explains that the non-diagonal gauge used in [18] is nothing but the standard *static* gauge in the Nambu action for the *dual* coordinate $\tilde{\alpha}$. Not imposing the conformal gauge allows one to have solutions consistent with the above static gauge choice.

Let us first note that the equation for α following from the action (2.8), i.e. $\partial_p(\sqrt{-g}g^{pq}D_q\alpha) = 0$, can be solved by setting

$$\sqrt{-g}g^{pq}D_q\alpha = -\epsilon^{pq}\partial_q\tilde{\alpha} \ , \qquad (2.15)$$

[2] We are grateful to N. Nekrasov for emphasizing to us the potential importance of applying T-duality in α.

where $\tilde{\alpha}$ should then satisfy

$$\partial_p(\sqrt{-g}g^{pq}\partial_q\tilde{\alpha}) + \epsilon^{pq}\partial_p C_q = 0 \ , \qquad \text{i.e.} \quad \partial_p(\sqrt{-g}g^{pq}\partial_q\tilde{\alpha}) = i\epsilon^{pq}\partial_p U_i^*\partial_q U_i \ . \quad (2.16)$$

Comparing (2.15) to (2.10) we observe that

$$\mathcal{J} = \int_0^{2\pi} \frac{d\sigma}{2\pi}\, \partial_1\tilde{\alpha} \ , \qquad \text{i.e.} \qquad \tilde{\alpha}(\tau, 2\pi) = \tilde{\alpha}(\tau, 0) + 2\pi\mathcal{J} \ , \tag{2.17}$$

which is satisfied, in particular, if one fixes the gauge by setting

$$\tilde{\alpha} = \mathcal{J}\sigma \ . \tag{2.18}$$

The limit of small $\tilde{\lambda}$ or large $\mathcal{J}$ (cf. (2.11)) is then the limit of large winding number of the dual coordinate $\tilde{\alpha}$. [3]

Let us now apply the 2-d duality systematically at the level of the string action (2.8). Replacing $D_p\alpha$ by $A_p + C_p$ where A_p is an auxiliary 2-d vector field, adding the "Lagrange multiplier" term $\epsilon^{pq}A_p\partial_q\tilde{\alpha}$, and then integrating out A_p we end up with the 2-d dual counterpart of (2.8)

$$\mathcal{L} = -\frac{1}{2}\sqrt{-g}g^{pq}(-\partial_p t\partial_q t + \partial_p\tilde{\alpha}\partial_q\tilde{\alpha} + D_p U_i^* D_q U_i) + \epsilon^{pq}C_p\partial_q\tilde{\alpha} \ , \tag{2.19}$$

where C_p was given in (2.14). Thus the "T-dual" background has no off-diagonal metric component but has a non-trivial NS-NS 2-form coupling in the $(\tilde{\alpha}, U_i)$ sector. Eliminating the 2-d metric g^{pq} we then get the Nambu form of the 2-d dual counterpart of the action (2.7),(2.8)

$$\mathcal{L} = \epsilon^{pq}C_p\partial_q\tilde{\alpha} \ - \sqrt{h} \ , \tag{2.20}$$

$$h = |\det h_{pq}| \ , \qquad h_{pq} = -\partial_p t\partial_q t + \partial_p\tilde{\alpha}\partial_q\tilde{\alpha} + D_{(p}U_i^* D_{q)}U_i \ . \tag{2.21}$$

If we now fix the static gauge

$$t = \tau \ , \qquad \tilde{\alpha} = \mathcal{J}\sigma \ , \tag{2.22}$$

we finish with the action equivalent to (2.13),(2.14)

$$I = J\int dt \int_0^{2\pi} \frac{d\sigma}{2\pi}\, \mathcal{L} \ , \qquad \mathcal{L} = C_0 - \sqrt{\tilde{h}} \ , \tag{2.23}$$

$$\tilde{h} = (1 + \tilde{\lambda}|D_1U_i|^2)(1 - |D_0U_i|^2) + \frac{1}{4}\tilde{\lambda}(D_0U_i^* D_1U_i + c.c.)^2 \ . \tag{2.24}$$

We thus uncover the origin of the string-theory counterpart of the WZ term C_0 in the spin-chain coherent state effective action: it comes from the 2-d NS-NS WZ term upon the static gauge fixing in the T-dual $\tilde{\alpha}$ action.

[3] A comment on the quantization condition. In the standard case of circular coordinate with radius a the dual coordinate has radius $\tilde{a} = \frac{\alpha'}{a^2}$. In the present case $\alpha' = \frac{1}{\sqrt{\lambda}}$ and $a = 1$ so the period of $\tilde{\alpha}$ should be $\frac{2\pi}{\sqrt{\lambda}}$. This implies that $J = \frac{\mathcal{J}}{\sqrt{\lambda}}$ should indeed be an integer winding number.

2.3 Eliminating time derivatives

The remaining steps are as in [16, 18]. We assume that higher powers of time derivatives are suppressed. To define a consistent $\frac{1}{\mathcal{J}^2} = \tilde{\lambda}$ expansion, we may then redefine the time coordinate so that the leading order approximation does not involve $\tilde{\lambda}$:

$$\tau \to \mathcal{J}^2 \tau = \tilde{\lambda}^{-1}\tau \ , \quad \text{i.e.} \quad t \to \tilde{\lambda}^{-1} t \ , \qquad \partial_0 \to \ \tilde{\lambda}\ \partial_0 \ , \tag{2.25}$$

thus getting the string action (2.23) with

$$\mathcal{L} = C_0 - \tilde{\lambda}^{-1}\sqrt{(1+\tilde{\lambda}|D_1 U_i|^2)(1-\tilde{\lambda}^2|D_0 U_i|^2) + \frac{1}{4}\tilde{\lambda}^3(D_0 U_i^* D_1 U_i + c.c.)^2} \ . \tag{2.26}$$

Expanding in powers of $\tilde{\lambda}$ (and omitting the constant term) this gives[4]

$$\mathcal{L} = \mathcal{L}_1 + \tilde{\lambda}\mathcal{L}_2 + O(\tilde{\lambda}^2) \ , \qquad \mathcal{L}_1 = -iU_i^*\partial_0 U_i - \frac{1}{2}|D_1 U_i|^2 \ , \tag{2.27}$$

$$\mathcal{L}_2 = \frac{1}{2}|D_0 U_i|^2 + \frac{1}{8}|D_1 U_i|^4 \ . \tag{2.28}$$

Finally, we should eliminate higher than first time derivatives in L by field redefinitions order by order in $\tilde{\lambda} = \frac{1}{\mathcal{J}^2}$. This was explained in detail in the $SU(2)$ case in [18]. For example, this amounts to eliminating $\partial_0 U_i$ from $\mathcal{L}_2$ using leading-order (Landau-Lifshitz type) equation for U_i following from $\mathcal{L}_1$. Note that the leading term $\mathcal{L}_1$ in the action (2.27) is the same as found by choosing the conformal gauge and eliminating α from the action using the constraints [21]. Also, in the next section we will again obtain an equivalent result by a different but related method which relies on the use of canonical transformations.

To summarize, the main lesson of the above derivation is that the dual angular coordinate $\tilde{\alpha}$ is just a replacement for the momentum P_α of the "fast" coordinate α. Applying T-duality $\alpha \to \tilde{\alpha}$ allows one to fix the static gauge $\tilde{\alpha} \sim \sigma$ which thus identifies the spin chain direction with $\tilde{\alpha}$ direction. Applied to $SU(3)$ sector the above procedure determines the subleading terms in the effective action. $\mathcal{L}_2$ in (2.28) may, in principle, be compared with the 2-loop effective action on the spin chain side generalizing the comparison in the $SU(2)$ case in [18]. Indeed, the corresponding 2-loop dilatation operator (for the $SU(3|2)$ superspin chain containing bosonic $SU(3)$ sector) was found in [30], and, as was recently pointed out in [31], the mixing of bosonic operators from the $SU(3)$ sector with the fermionic operators at the two (and higher) loop order is effectively suppressed in the long spin chain (large $L = J$) limit.

[4]In the $SU(2)$ case one finds [18] $\mathcal{L} = C_0 - \frac{1}{8}(\partial_1 n_i)^2 + \frac{\tilde{\lambda}}{8}\left[(\partial_0 n_i)^2 + \frac{1}{16}(\partial_1 n_i)^4\right] - \frac{\tilde{\lambda}^2}{32}\left[(\partial_0 n_i \partial_1 n_i)^2 - \frac{1}{2}(\partial_0 n_i)^2(\partial_1 n_k)^2 + \frac{1}{32}(\partial_1 n_i \partial_1 n_i)^3\right] + O(\tilde{\lambda}^3)$.

3. String theory side: general fast motion ($SO(6)$ sector)

Now we can proceed to analyze the $SO(6)$ sector. On the string side we have a string that moves almost at the speed of light[5] along the S^5. It is then natural to start by isolating a coordinate that describes the fast motion in order to use the approximation that the velocities of all *other* coordinates are small. In previous work [16, 18, 20, 21][6] this was achieved by means of an appropriate change of coordinates. However, this can be done only if we already know the particular string configuration we are aiming to describe. Here, instead, we would like to isolate a fast coordinate independently of the type of solution, i.e. in a universal way that will apply to rotating [7, 32, 11], pulsating [24, 15] and other similar solutions (see *e.g.* [33, 34]) that describe fast moving strings on S^5.

The common feature to all of them is that each piece of the string is moving along a maximum circle almost at the speed of light. Since each point of the string moves, in general, along a different massless geodesic in $R_t \times S^5$, to define the fast coordinate we need to know the position *and* velocity of each point of the string. Therefore, we can isolate the fast coordinate only if we use the *phase space* description.

What we find is that the circle along which each piece of string moves, slowly changes in time due to interactions with its neighbors along the string (represented by the terms containing sigma derivatives). This is in exact agreement with the picture on the spin chain side. In an appropriate gauge, each piece of string moving along a circle carries one unit of R-charge and corresponds to a half-BPS operator on the spin chain carrying the same charge. On the spin chain, the operator we have at each site (given, for example, by the mean value of the spin in the $SU(2)$ case) also changes in time precisely as a result of the interaction with its neighbors. Notice that here we are mapping a time dependent classical string into a time dependent coherent state on the spin chain side and not into an energy eigenstate. Energy eigenstates of the spin chain or SYM theory should correspond to single-string eigenstates in the bulk, while classical string solutions should be represented by coherent states.

In the rest of this section we do the following steps. First, we isolate the fast coordinate which we shall call α as in section 2. Then we obtain the leading order terms in the action by considering the limit in which α changes much faster than all other coordinates. One problem is that the Lagrangian contains terms which oscillate in α (proportional to $e^{\pm 2i\alpha}$). Since they average to zero, at leading order they can be discarded. However, when we go to higher orders, they give a contribution. The simplest way to treat them is to eliminate them order by order using coordinate trans-

[5]The corresponding $\tilde{\lambda} \to 0$ limit can also be seen as a small tension limit [25].

[6]See also [27] for a different approach.

formations. In this way we end up with a Lagrangian that, to the order considered, is independent of the fast coordinate, i.e., like (2.8), has an isometry $\alpha \to \alpha +$ constant. We can then perform a T-duality $\alpha \to \tilde{\alpha}$ and fix a static gauge as in (2.18),(2.22), i.e. $\tilde{\alpha} = \ell\sigma$, where $\ell = \frac{L}{\sqrt{\lambda}}$ is essentially the momentum conjugate to α. We end up with the standard Nambu action with an extra term due to the presence of a B_{mn}–field. This Wess-Zumino type term ensures the correct phase space structure of the action. After that the large energy or large L expansion is simply a Taylor expansion of the square root in the Nambu action.

3.1 Isolating the "fast" angular coordinate α

Consider the Lagrangian for the string on $R \times S^5$ written in terms of AdS_5 time t and 6 real coordinates X_m

$$\mathcal{L} = -\frac{1}{2}[-(\partial_p t)^2 + (\partial_p X_m)^2 + \Lambda(X_m X_m - 1)] , \qquad m = 1, \ldots, 6 , \quad p = 0, 1 . \tag{3.1}$$

The corresponding conformal gauge constraints are (we choose $t = \kappa\tau$ as an additional gauge fixing condition)

$$(\partial_0 X_m)^2 + (\partial_1 X_m)^2 = \kappa^2 , \qquad \partial_0 X_m \partial_1 X_m = 0 . \tag{3.2}$$

In the simplest case when X_m does not depend on σ, i.e. the string is point-like, we get the geodesic equations $\partial_0^2 X_m + \Lambda X_m = 0$, $\Lambda = (\partial_0 X_m)^2 = \kappa^2$. These are solved by

$$X_m(\tau) = a_m \cos\alpha + b_m \sin\alpha , \qquad \alpha = \kappa\tau , \quad a_m^2 = 1 , \quad b_m^2 = 1 , \quad a_m b_m = 0, \tag{3.3}$$

or, equivalently, by

$$X_m = \frac{1}{\sqrt{2}} \left(e^{i\alpha} V_m + e^{-i\alpha} V_m^* \right) , \qquad V_m = \frac{a_m - i b_m}{\sqrt{2}} , \tag{3.4}$$

where

$$V_m V_m^* = 1 , \qquad V_m V_m = 0 , \qquad V_m^* V_m^* = 0 . \tag{3.5}$$

The constant 6-vectors a_m and b_m parametrize the space of massless geodesics in $R \times S^5$, *i.e.* the space of maximum circles in S^5. Equivalently, they parameterize the Grassmanian $G_{2,6}$ – the space of planes in R^6 that go through the origin.[7] Thus, as was noted in [27], the coset $G_{2,6} = SO(6)/(SO(4) \times SO(2)) = SU(4)/S(U(2) \times U(2))$ is the moduli

[7] In general, the Grassmanian $G_{k,n}$ is the space of all k dimensional hyperplanes of the vector space R^n.

space of geodesics in S^5. On the SYM side the same space appeared as a natural coherent state space for $SO(6)$ spin chain in [21]. Let us mention also that V_m describing the Grassmanian may be interpreted as coordinates of CP^5 subject to an additional condition $V_m^2 = 0$, i.e. they define an 8-dimensional subspace of $CP^5 = SU(6)/U(5)$.

If we now consider an extended relativistic closed string, the trajectory of each piece of the string can be described as a circular trajectory but with the parameters a_m and b_m (or V_m) which determine its orientation, slowly changing in time $t = \kappa\tau$ and in σ (the coordinate along the string). Also, the speed along the circle is time dependent in this case. A familiar analogy is an orbital motion of a planet in a solar system. It moves along an ellipse around the Sun but the parameters of the ellipse slowly change in time due to perturbations by other planets, moons, etc. In both cases, the fast coordinate is an angle in the plane of motion determined by the position and the velocity.

To obtain the leading order result for an effective Lagrangian of "slow" coordinates it is sufficient to work in the conformal gauge and use again the additional gauge condition $t = \kappa\tau$. Then the phase space Lagrangian is simply

$$\mathcal{L} = -\frac{1}{2}\kappa^2 + P_m\dot{X}_m - \frac{1}{2}P_mP_m - \frac{1}{2}X'_mX'_m - \frac{1}{2}\Lambda(X_mX_m - 1) \ . \tag{3.6}$$

Here and below we define the momenta (P_m, etc.) as derivatives of the Lagrangian, i.e. do not include the tension $\sqrt{\lambda}$ factor in the action (2.7) in their definition. To isolate the coordinate α we introduce its conjugate momentum P_α and do the following coordinate transformation

$$\begin{aligned} X_m &= \cos\alpha \ a_m + \sin\alpha \ b_m \ , \\ P_m &= P_\alpha(-\sin\alpha \ a_m + \cos\alpha \ b_m) \ , \end{aligned} \tag{3.7}$$

where, as in (3.3), $a_m^2 = 1$, $b_m^2 = 1$, $a_mb_m = 0$. This can be conveniently written as[8]

$$\begin{aligned} X_m &= \frac{1}{\sqrt{2}}\left(e^{i\alpha}V_m + e^{-i\alpha}V_m^*\right) \ , \\ P_m &= \frac{1}{\sqrt{2}}iP_\alpha\left(e^{i\alpha}V_m - e^{-i\alpha}V_m^*\right) \ , \end{aligned} \tag{3.8}$$

where as in (3.5)

$$V_m = \frac{a_m - ib_m}{\sqrt{2}} \ , \qquad V_mV_m^* = 1 \ , \qquad V_m^2 = 0 \ . \tag{3.9}$$

[8]For constant α, this is actually quite analogous to the usual transformation $(q,p) \to (a,a^*)$ to "coherent coordinates" for a harmonic oscillator.

The constraints on V_m ensure that $X_m^2 = 1$ and $P_m X_m = 0$ (which is a consequence of $X^2 = 1$ and $P_m = \dot{X}_m$ in the conformal gauge). There is a redundant degree of freedom which is obvious from the gauge invariance ($\beta = \beta(\sigma, \tau)$ is an arbitrary function):

$$\alpha \to \alpha - \beta \ , \qquad V_m \to e^{i\beta} V_m \ , \qquad V_m^* \to e^{-i\beta} V_m^* \ . \tag{3.10}$$

We could fix this gauge invariance by choosing *e.g.* $a_6 = 0$ or $V_6 + V_6^* = 0$, but it is more convenient not to do this to preserve the $SO(6)$ rotational symmetry. It is then natural to introduce the covariant derivatives defined by

$$D_p\alpha = \partial_p \alpha + C_p \ , \quad D_p V_m = \partial_p V_m - iC_p V_m \ , \quad D_p V_m^* = \partial_p V_m^* + iC_p V_m^* \ , \tag{3.11}$$

where as in (2.5) the $U(1)$ gauge field C_p is not an independent variable but is given by

$$C_p = -iV_m^* \partial_p V_m \ . \tag{3.12}$$

Now it is easy to derive the following useful identities[9]

$$\begin{aligned} &P_m \partial_p X_m = P_\alpha D_p \alpha \ , \qquad\qquad P_m^2 = P_\alpha^2 \ , \\ &X_m'^2 = (D_1\alpha)^2 + |D_1 V|^2 + \frac{1}{2}\left[e^{2i\alpha}(D_1 V)^2 + e^{-2i\alpha}(D_1 V^*)^2\right] \ , \end{aligned} \tag{3.13}$$

which allow us to rewrite the Lagrangian (3.6) as

$$\begin{aligned} \mathcal{L} = &-\frac{1}{2}\kappa^2 + P_\alpha D_0\alpha - \frac{1}{2}P_\alpha^2 - \frac{1}{2}\left\{(D_1\alpha)^2 + |D_1 V|^2 + \frac{1}{2}\left[e^{2i\alpha}(D_1 V)^2 + e^{-2i\alpha}(D_1 V^*)^2\right]\right\} \\ &-\mu_1(|V|^2 - 1) - \frac{1}{2}\left(\mu_2 V^2 + c.c.\right) \ , \end{aligned} \tag{3.14}$$

where μ_1 is a real and μ_2 is a complex Lagrange multiplier field. The conformal gauge constraints are then

$$\begin{aligned} &P_m P_m + X_m' X_m' = P_\alpha^2 + (D_1\alpha)^2 + |D_1 V|^2 + \frac{1}{2}\left[e^{2i\alpha}(D_1 V)^2 + e^{-2i\alpha}(D_1 V^*)^2\right] = \kappa^2 \ , \\ &P_m X_m' = P_\alpha D_1 \alpha = 0 \ . \end{aligned} \tag{3.15}$$

Notice that we could use that $(D_1 V)^2 = (\partial_1 V)^2$ (since $V^2 = 0$ and $V\partial_1 V = 0$) to simplify these expressions but we prefer to keep the $U(1)$ gauge invariance (3.10) manifest.

Another point we note for later reference is that the second constraint in (3.15) implies $D_1\alpha = \partial_\sigma\alpha + C_1 = 0$. Since the string is closed, the coordinates in (3.8) must be periodic in σ. We may assume then that $\alpha(\tau, \sigma + 2\pi) = \alpha(\tau, \sigma)$ (an a priori possible

[9] We shall often omit index m on V_m in quadratic relations assuming summation over m.

"winding" part can be absorbed into V_m, cf. (3.10)). Then we obtain the following constraint

$$0 = \int_0^{2\pi} \frac{d\sigma}{2\pi} \partial_\sigma \alpha = - \int_0^{2\pi} \frac{d\sigma}{2\pi} C_1 = i \int_0^{2\pi} \frac{d\sigma}{2\pi} V_m^* \partial_\sigma V_m \ . \tag{3.16}$$

On the field theory side, the fact that the string is closed is related to the fact that the dual operators are given by the single $SU(N)$ trace which is invariant under cyclic permutations of local fields under the trace. In the spin chain description such operators are represented by states of a closed chain which are invariant under cyclic permutations. This invariance gives rise to a condition equivalent to (3.16) (see section 4).

3.2 Leading order in large energy expansion

We can take into account that α is a fast variable by setting

$$\alpha = \kappa\tau + \bar{\alpha} \ , \qquad\qquad P_\alpha = \kappa + p_\alpha \ ,$$

where $\bar{\alpha}$ and p_α are the new variables, and then taking the limit $\kappa \to \infty$ while keeping $\kappa \dot{V}$, $\kappa \dot{\bar{\alpha}}$ and κp_α fixed.[10] The only difference with the previous rotating string cases is that here there are terms in (3.14),(3.15) proportional to $e^{\pm i\kappa\tau}$. It is clear that for large κ these terms can be ignored since they average to zero.[11] Then, the conformal gauge constraints (3.15) determine $\bar{\alpha}' = \alpha'$ ($\equiv \partial_1 \alpha$) and p_α as (to leading order in $\frac{1}{\kappa}$):

$$D_1 \alpha = 0 \quad \Rightarrow \quad \alpha' = -C_1 = i V^* \partial_1 V \ , \qquad\qquad \kappa p_\alpha = -\frac{1}{2} D_1 V D_1 V^* \ . \tag{3.17}$$

Using that $D_1 \alpha = 0$ in the Lagrangian (3.14) and taking the same $\kappa \to \infty$ limit we find

$$\mathcal{L} = \kappa \dot{\bar{\alpha}} - i\kappa V^* \dot{V} - \frac{1}{2} D_1 V D_1 V^* - \mu_1 (|V|^2 - 1) - \frac{1}{2} (\mu_2 V^2 + c.c.) \ . \tag{3.18}$$

The first term here is a total derivative and may be omitted, so that the corresponding action takes the form similar to (2.23),(2.27)

$$I = L \int dt \int_0^{2\pi} \frac{d\sigma}{2\pi} \ \mathcal{L}_1 \ , \qquad L = \sqrt{\lambda}\ell \ , \quad \ell \equiv \frac{1}{\sqrt{\tilde{\lambda}}} \simeq \kappa \simeq P_\alpha \ , \tag{3.19}$$

$$\mathcal{L}_1 = -i V^* \dot{V} - \frac{1}{2} |D_1 V|^2 \ , \tag{3.20}$$

[10]See [26, 27] for related ideas. This limit is also reminiscent of the so called wrapped or non-relativistic limit [35].

[11]An equivalent averaging was also considered in [27].

where in (3.20) we rescaled the time coordinate as in (2.25) (and did not explicitly include the Lagrange multiplier terms).

We thus find the "Landau-Lifshitz" version of the $G_{2,6}$ sigma model, i.e. with the first time derivative WZ-type term instead of the usual quadratic time-derivative term.[12] This Lagrangian (3.18) is the same as found in [27] through a different procedure.[13]

A Lagrangian equivalent to (3.18) was found also on the spin chain side in [21] (see also the next section for the derivation). As was shown in [21], the corresponding Hamiltonian represents a low-energy (continuum) limit of the expectation value of the $SO(6)$ spin chain Hamiltonian of [9] in the natural $SO(6)$ coherent state $|v\rangle$ obtained by applying the $SO(6)$ transformation from $G_{2,6} = SO(6)/[SO(4) \times SO(2)]$ to the BPS ground state $|0\rangle = (0, 0, 0, 0, 1, i)$ at each site of the spin chain. This state may be parametrized by a real 6×6 antisymmetric matrix $\mathrm{m}_{mn} = \langle v|M_{mn}|v\rangle$, where M_{mn} is a hermitian $SO(6)$ generator, satisfying the constraints [21] $\mathrm{m}^3 = \mathrm{m}$, $\mathrm{tr}(\mathrm{m}^2) = 2$. We can see the relation to the present discussion if we solve these constraints by introducing a complex 6-vector v_m subject to $|v|^2 = 1$, $v^2 = 0$, with $\mathrm{m}_{mn} = i(v_m v_n^* - v_n v_m^*)$. Going back to the string theory side, let us note that the $SO(6)$ generators corresponding to the Lagrangian (3.1) written in terms of X_m become, after using (3.8),

$$M_{mn} = \sqrt{\lambda}\int_0^{2\pi} \frac{d\sigma}{2\pi}(X_m P_n - X_n P_m) \simeq iL\int_0^{2\pi} \frac{d\sigma}{2\pi}(V_n V_m^* - V_m V_n^*) + \dots \ , \quad (3.21)$$

where we used that $L \simeq \sqrt{\lambda}P_\alpha$. This is the same result as found for the coherent state expectation value of the rotation generator on the spin chain side [21], implying the identification between the corresponding complex vector variables v_m and V_m.

The equivalence between the string and spin chain effective actions explains and generalizes the previous results about the agreement between the leading-order terms in the energies of pulsating/rotating strings and dimensions of the corresponding gauge theory operators which were found using the Bethe ansatz approach [24, 10, 15, 31]. In addition to its universality, a bonus of the coherent state approach [16], on which we shall elaborate below, is that it explicitly relates the form of the string solution to the structure of the corresponding coherent operator on the gauge theory side. We shall further clarify and illustrate this point in sections 4 and 5 by considering particular examples of pulsating solutions. But first let us generalize the above discussion and

[12] Note that this model is integrable since it is obtained as a limit of an integrable $SO(6)/SO(5)$ sigma model (in general, Grassmanian cosets are classically integrable [36]).

[13] One difference compared to [27] is that we write $\mathcal{L}$ in $U(1)$ gauge invariant form. The expression in [27] should be considered as a gauge fixed version of (3.18).

show how we can obtain, in a systematic fashion, higher order corrections to the action (3.19).

3.3 Elimination of the fast variable and higher order terms

If we go to higher orders, working in the conformal gauge introduces a problem pointed out in [18]: in the present case it is related to the fact that P_α is identified (to leading order) with $\kappa = \frac{E}{\sqrt{\lambda}}$. The expansion of the energy will therefore contain powers of the energy itself. We would prefer to associate P_α to a conserved momentum (like J in the rotating case or L in the pulsating one). While in the conformal gauge this is the case at the leading order this is not so at the higher orders of large energy expansion. Furthermore, we want such conserved momentum to be uniformly distributed along the string (i.e. do not depend on σ) so that we can establish a simple correspondence with the spin chain side.

The solution to this problem is the same as in [18]. Instead of fixing the conformal gauge as we did in the previous subsection, we can use the phase space description without fixing the world-sheet metric g_{pq}. Then, fixing the partial gauge $t = \tau$ (here we prefer not to explicitly include the factor of κ), we get the following phase space Lagrangian (cf. (3.6)):

$$\mathcal{L} = -P_t + P_m \dot{X}_m - \frac{1}{2\sqrt{-g}g^{00}} \left(-P_t^2 + P_m P_m + X_m' X_m'\right) - \frac{g^{01}}{g^{00}} P_m X_m' - \frac{1}{2}\Lambda \left(X_m X_m - 1\right) . \tag{3.22}$$

Doing the same transformation as in (3.8) we obtain the Lagrangian in terms of the new variables (cf. (3.14))

$$\begin{aligned} \mathcal{L} = & -P_t + P_\alpha D_0 \alpha - \frac{1}{2\sqrt{-g}g^{00}} \Big(- P_t^2 + P_\alpha^2 + (D_1\alpha)^2 + |D_1 V|^2 \\ & + \frac{1}{2}\left[e^{2i\alpha}(D_1 V)^2 + e^{-2i\alpha}(D_1 V^*)^2\right] \Big) - \frac{g^{01}}{g^{00}} P_\alpha \, D_1 \alpha \\ & - \mu_1 (V V^* - 1) - \frac{1}{2}\left(\mu_2 V^2 + \mu_2^* V^{*2}\right) . \end{aligned} \tag{3.23}$$

The equations of motion for the 2-d metric components determine P_t (the momentum conjugate to the AdS_5 time coordinate t) as a function of the other variables, and also imply that $D_1\alpha = 0$. Setting $D_1\alpha = 0$ in (3.23) we obtain

$$\begin{aligned} \mathcal{L} = & -P_t + P_\alpha \dot{\alpha} - i P_\alpha V^* \dot{V} - \frac{1}{2\sqrt{-g}g^{00}} \Big(- P_t^2 + P_\alpha^2 + |D_1 V|^2 \\ & + \frac{1}{2}\left[e^{2i\alpha}(D_1 V)^2 + e^{-2i\alpha}(D_1 V^*)^2\right] \Big) - \mu_1 (V V^* - 1) - \frac{1}{2}\left(\mu_2 V^2 + c.c.\right) \end{aligned} \tag{3.24}$$

We see that P_α is indeed the momentum conjugate to α. Also, the momentum conjugate to V_n is then

$$P_{V_n} = -iP_\alpha V_n^* \ .$$

Thus, V and V^* are not canonically conjugate variables since, in general, P_α is not constant. For later reference let us list also the non-vanishing Poisson brackets

$$\begin{aligned} [\alpha(\sigma,\tau), P_\alpha(\sigma',\tau)] &= \delta(\sigma-\sigma') \ , \\ [\alpha(\sigma,\tau), V_n^*(\sigma',\tau)] &= [\alpha(\sigma,\tau), \frac{iP_{V_n}(\sigma',\tau)}{P_\alpha(\sigma',\tau)}] = -\frac{V_n^*(\sigma,\tau)}{P_\alpha(\sigma,\tau)}\delta(\sigma-\sigma') \ , \\ [V_m(\sigma,\tau), V_n^*(\sigma',\tau)] &= [V_m(\sigma,\tau), \frac{iP_{V_n}(\sigma',\tau)}{P_\alpha(\sigma',\tau)}] = \frac{i}{P_\alpha(\sigma,\tau)}\delta_{mn}\delta(\sigma-\sigma') \ . \end{aligned} \tag{3.25}$$

Let us recall that we wanted to associate P_α with a large conserved momentum which characterizes the solutions but, due to the α dependence of the Lagrangian (3.24), the equation of motion for α implies that P_α is not conserved. To identify the relevant conserved quantity let us do a coordinate transformation that preserves (up to a total time derivative) the form of the terms $P_\alpha(\dot\alpha - iV^*\dot V)$ in (3.24) but at the same time eliminates all α dependence in the rest of the Lagrangian. The new P_α will be conserved and can be used to classify the solutions. Unfortunately, it is not clear to us if this change of coordinates can be done in a closed manner but it can certainly be done order by order in an expansion in $\frac{1}{P_\alpha}$ as we explain below.

Even perturbatively, finding this transformation directly is hopelessly lengthy if we do not resort to the method of canonical perturbation theory which we review in Appendix following [37]. The infinitesimal canonical transformation is determined by a function $W(\bar y)$ through

$$\delta\bar y = \bar y_{\rm new} - \bar y_{\rm old} = [\bar y, W(\bar y)] \ , \qquad \bar y \equiv (\alpha, P_\alpha, V, V^*) \ , \tag{3.26}$$

where we used the (non-canonical) Poisson brackets of eq.(3.25).[14] We cannot use an arbitrary function W since we want, at the same time, to preserve the constraint $V^2 = 0$, $V^{*2} = 0$ but generically $[V^2, W] \neq 0$. Instead of restricting the function W we can define a different transformation using the Dirac bracket[15] since then the constraints commute with all functions. The definition of this bracket is

$$[f,g]_D = [f,g] - \frac{iP_\alpha}{4}[f,V^2][V^{*2},g] + \frac{iP_\alpha}{4}[f,V^{*2}][V^2,g] \ . \tag{3.27}$$

[14] The fact that our variables are non-canonical is not a problem for applying the perturbation theory [38].

[15] This is equivalent to adding to δV a term proportional to V^* which trivially preserves the terms linear in time derivatives and can be chosen such that $\delta(V^2) = 0$.

Then using (3.25) we find the following non-vanishing brackets

$$
\begin{aligned}
[\alpha(\sigma,\tau), P_\alpha(\sigma',\tau)]_{_D} &= \delta(\sigma-\sigma')\ , \\
[\alpha(\sigma,\tau), V_n^*(\sigma',\tau)]_{_D} &= -\frac{V_n^*(\sigma,\tau)}{P_\alpha(\sigma,\tau)}\delta(\sigma-\sigma')\ , \\
[V_m(\sigma,\tau), V_n^*(\sigma',\tau)]_{_D} &= \frac{i}{P_\alpha(\sigma,\tau)}\Big[\delta_{mn} - V_m^*(\sigma,\tau)V_n(\sigma,\tau)\Big]\delta(\sigma-\sigma')\ .
\end{aligned}
\tag{3.28}
$$

Note that these brackets are of order $\frac{1}{P_\alpha}$ in the sense that the right hand side contains one less power of P_α than the left hand side. This means that, upon quantization (which would require also to include fermionic terms), the commutators will vanish in the limit $P_\alpha \to \infty$ which can then be understood as a classical limit.[16]

We are still to satisfy the constraint $VV^* = 1$, but this constraint is preserved if

$$
[VV^*, W]_{_D} = \frac{1}{P_\alpha}\left(\frac{\delta W}{\delta\alpha} + iV^*\frac{\delta W}{\delta V} - iV\frac{\delta W}{\delta V}\right) = 0\ , \tag{3.29}
$$

which expresses the fact that W has to be gauge invariant, i.e. invariant under (3.10).

Let us now perform the transformation (3.26) on the Lagrangian (3.24). In principle, we need a finite transformation, but if we expand it in powers of $\frac{1}{P_\alpha}$ then, at lowest order, we only need an infinitesimal transformation generated by some $W = W_1$ (we explain this in more detail in Appendix, see eqs. (A.16) and (A.17)). To transform the Lagrangian we note that a generic function $f(\bar{y})$ transforms in such a way that

$$
f_{\text{new}}(\bar{y}_{\text{new}}) = f_{\text{old}}(\bar{y}_{\text{old}}) \quad\Rightarrow\quad f_{\text{new}} = f_{\text{old}} + \frac{\partial f_{\text{old}}}{\partial \bar{y}_j}\,[W_1, \bar{y}_j]_{_D} = f_{\text{old}} + [W_1, f_{\text{old}}]_{_D}. \tag{3.30}
$$

Note the opposite order of W and $\bar{y}$ in the Dirac bracket as compared to (3.26). Since P_t, the terms linear in derivatives in (3.24), and the constraints, are all invariant under such transformations, then the only terms which transform are

$$
\mathcal{L}_0 = P_\alpha^2\ , \quad \langle\mathcal{L}_1\rangle = |D_1 V|^2\ , \quad \{\mathcal{L}_1\} = \frac{1}{2}\left[e^{2i\alpha}(D_1V)^2 + e^{-2i\alpha}(D_1V^*)^2\right]\ , \tag{3.31}
$$

which appear in the Lagrangian (3.24) multiplied by a common factor which is invariant. Here $\langle ...\rangle$ denotes terms averaged over α and therefore α-independent, and $\{...\}$ denotes terms which are oscillating in α and thus average to zero.

---

[16] It also follows that in the quantum treatment V^* should roughly be identified with $\frac{\delta}{\delta V}$ which makes the structure of the Hamiltonian very reminiscent of the form of the dilatation operator discussed in [39] in terms of the scalar field Φ_i and $\check{\Phi}_i$ defined as an operator that "destroys" a scalar field.

The leading term in the large P_α expansion is $\mathcal{L}_0$, so if we want to cancel the α dependence at the lowest order, we need a function W_1 such that

$$[W_1, \mathcal{L}_0]_D = [W_1, P_\alpha^2]_D = 2P_\alpha \frac{\delta W}{\delta\alpha} = -\{\mathcal{L}_1\} = -\frac{1}{2}\left[e^{2i\alpha}(D_1V)^2 + e^{-2i\alpha}(D_1V^*)^2\right] . \tag{3.32}$$

Integrating this equation, we obtain W_1

$$W_1 = \frac{i}{8}\int d\sigma\, \frac{1}{P_\alpha}\left(e^{2i\alpha}(D_1V)^2 - e^{-2i\alpha}(D_1V^*)^2\right) , \tag{3.33}$$

which is (manifestly) gauge invariant and so preserves the constraint $VV^* = 1$ as required. We have used that $D_1\alpha = 0$. The transformed Lagrangian, written in the new variables, is the same as the original one written in the old variables, except that the term $\{\mathcal{L}_1\}$ is no longer present. The end result of this procedure is thus equivalent to averaging the Lagrangian over α as we did in the previous subsection.

To go beyond the leading order we use the results described in Appendix (see eq.(A.17)) and compute the second order term as

$$\bar{\mathcal{L}}_2 = \frac{1}{2}\,\langle\ [W_1, \{\mathcal{L}_1\}]_D\ \rangle , \tag{3.34}$$

where, as before, $\langle ...\rangle$ denotes averaging over α. This follows from the second order contribution of W_1 to the transformation. There is also a W_2 contribution to W at this order but it just cancels the α dependence in the second order terms, as W_1 did for the first order ones. Therefore, what we need to compute is

$$\begin{aligned}&\left\langle \frac{i}{32}\left[\int d\sigma \frac{1}{P_\alpha}\left(e^{2i\alpha}(D_1V)^2 - e^{-2i\alpha}(D_1V^*)^2\right), \int d\sigma' \left(e^{2i\alpha}(D_1V)^2 + e^{-2i\alpha}(D_1V^*)^2\right)\right]_D \right\rangle \\ &= -\frac{1}{4P_\alpha^2}\left(D_1^2VD_1^2V^* - (V^*D_1^2V)(VD_1^2V^*) - \frac{3}{2}(D_1V)^2(D_1V^*)^2\right) ,\end{aligned} \tag{3.35}$$

where we used that $D_1\alpha = 0$ and anticipated that we are going to choose a gauge where P_α =const by taking $D_1P_\alpha = \partial_1 P_\alpha = 0$. Replacing in the Lagrangian we get

$$\begin{aligned}\mathcal{L} = &-P_t + \dot{\alpha}P_\alpha - iP_\alpha V^*\dot{V} - \frac{1}{2\sqrt{-g}g^{00}}\Big[- P_t^2 + P_\alpha^2 + (D_1V)(D_1V)^* \\ &-\frac{1}{4P_\alpha^2}\left(D_1^2VD_1^2V^* - (V^*D_1^2V)(VD_1^2V^*) - \frac{3}{2}(D_1V)^2(D_1V^*)^2\right) + \mathcal{O}(\frac{1}{P_\alpha^3})\Big] \\ &-\mu_1(VV^*-1) - \frac{1}{2}\left(\mu_2V^2 + c.c.\right) .\end{aligned} \tag{3.36}$$

Now we can choose a gauge $P_\alpha = \ell$=const as in [18] (here ℓ is a generalization of $\mathcal{J}$ in (2.11), i.e. $L = \sqrt{\lambda}\ell$ as in (3.19)) and substitute the expression for P_t from the

constraint following from variation over g^{00}, or do the T-duality transformation $\alpha \to \tilde{\alpha}$ and fix the static gauge $\tilde{\alpha} = \ell\sigma$ as discussed in section 2. In both cases we get the following Lagrangian (cf. (2.23))

$$\mathcal{L} = -i\ell V^* \dot{V} - \sqrt{h} - \mu_1(VV^* - 1) - \frac{1}{2}\left(\mu_2 V^2 + c.c.\right) \ , \tag{3.37}$$

$$h = \ell^2 + |D_1 V|^2 - \frac{1}{4\ell^2}\left(D_1^2 V D_1^2 V^* - (V^* D_1^2 V)(V D_1^2 V^*) - \frac{3}{2}(D_1 V)^2 (D_1 V^*)^2\right) + \mathcal{O}(\frac{1}{\ell^3}) \ . \tag{3.38}$$

The corresponding action takes the same form as (3.19) which generalizes (2.23). Expanding for large $\ell \equiv \frac{1}{\sqrt{\tilde{\lambda}}}$ is now a straightforward Taylor expansion of $\sqrt{h}$. Notice, however, that we can only do the expansion up to order $\frac{1}{\ell^3}$ since we did not compute further terms in the canonical transformation. The final result for the first two leading terms in the effective Lagrangian is thus (we omit a constant order ℓ term)

$$\begin{aligned}\mathcal{L} = \ell\Big[&- iV^*\dot{V} - \frac{1}{2\ell^2} D_1 V D_1 V^* \\ &+ \frac{1}{8\ell^4}\Big((|D_1 V|^2)^2 + |D_1^2 V|^2 - |V^* D_1^2 V|^2 - \frac{3}{2}|(D_1 V)^2|^2\Big)\Big] \\ &- \mu_1(VV^* - 1) - \frac{1}{2}\left(\mu_2 V^2 + \mu_2^* V^{*2}\right) \ .\end{aligned} \tag{3.39}$$

In the special case of $SU(2)$ sector of string states with two angular momenta this Lagrangian can be seen to agree with the Lagrangian found in [18] (and also with (2.28) in $SU(3)$ sector, upon using leading-order equations there to eliminate the time derivative terms). One can also check that this Lagrangian does reproduce the large ℓ expansion of the exact pulsating solution discussed in section 6.

At this order the Hamiltonian follows from (3.39) as (we now include the constant term ℓ):

$$H = \ell + \int_0^{2\pi} \frac{d\sigma}{2\pi}\left\{\frac{1}{2\ell^2}|D_1 V|^2 + \frac{1}{8\ell^4}\Big((|D_1 V|^2)^2 + |D_1^2 V|^2 - |V^* D_1^2 V|^2 - \frac{3}{2}|(D_1 V)^2|^2\Big)\Big]\right\} \tag{3.40}$$

If we find a solution of the equations of motion for this Hamiltonian, or equivalently the Lagrangian (3.39), we can obtain the time evolution of α as

$$\alpha = \frac{\partial H}{\partial \ell} t = t + \omega_\alpha t \ , \tag{3.41}$$

where we denoted the (small) correction as $\omega_\alpha = \partial(H - \ell)/\partial\ell$. If we now want to know the profile of the string $X_m(\sigma, t)$ we should first undo the canonical transformation

(3.26) and then use (3.8). This results in

$$X_m = \frac{1}{\sqrt{2}} \left\{ e^{it+i\omega_\alpha t} \left[V_m - \frac{i}{\ell} \left(\delta_{mn} - V_m^* V_n - V_m V_n^* \right) \frac{\delta W_1}{\delta V_n^*} \right] + \text{c.c.} \right\} \ , \tag{3.42}$$

where we can further use that (see (3.33))

$$\frac{\delta W_1}{\delta V_n^*} = \frac{i}{4\ell} e^{-2i\alpha} D_1^2 V_n^* \ . \tag{3.43}$$

As follows from the discussion in the next section, it is natural to assume that the transformed variables are identified with similar variables on the spin chain side. If that is so,[17] (3.42) can be understood as a change of variables from the spin chain variables $V_m(\sigma, t)$ to the string configuration $X_m(\sigma, t)$. At leading order we see from (3.42) that the transformation is simply (3.8), the first correction being of order ℓ^{-2}.

As a final comment we note that to obtain the expression (3.42) we expanded the exponent in

$$e^{i\alpha - i[\alpha, W_1]} \simeq e^{i\alpha}(1 - i[\alpha, W_1] + \ldots) \tag{3.44}$$

and used the property (3.29). This expansion is valid at this order, but, as discussed before, further expanding

$$e^{it+i\omega_\alpha t} \simeq e^{it}(1 + i\omega_\alpha t + \ldots) \tag{3.45}$$

would be incorrect since secular terms, *i.e.* terms linear in t, would appear.

4. Field theory side: effective action from $SO(6)$ spin chain

Our general aim is to compare the energies of strings moving fast in the S^5 part of $AdS_5 \times S^5$ with the anomalous dimensions of the corresponding operators in the scalar sector of $\mathcal{N} = 4$ SYM theory. The operators we are interested in can be written as

$$\mathcal{O} = C_{m_1 \ldots m_L} \text{tr} \left(X^{m_1} \ldots X^{m_L} \right) \ , \tag{4.1}$$

where X_m ($m = 1, \ldots, 6$) are real scalar fields in adjoint representation of $SU(N)$ and the coefficients $C_{m_1 \ldots m_L}$ are complex in general. If we just wanted operators that have a well defined anomalous dimension, namely eigenvectors of the dilatation operator, then, since $\mathcal{O}^\dagger$ has the same conformal dimension as $\mathcal{O}$, we could always choose the coefficients $C_{m_1 \ldots m_L}$ to be real. However, this is not possible if, for example, we want to find operators of given R-charges such as, *e.g.*, $\text{tr} X^J$ where $X = X_1 + iX_2$. Also, as

[17] We are grateful to A. Mikhailov for a question on this issue.

described in [16, 18, 21], in our case we are not looking for operators which are eigenstates of the dilatation operator but for "coherent" operators that behave under the renormalization group evolution as the classical string does under the time evolution. This can be understood more easily if we do a conformal mapping of the field theory to $R \times S^3$. In that case the operators we are looking for create coherent states which are time dependent in exactly the same way as the classical string solutions are. So the conclusion is that, for our purpose, we cannot restrict the coefficients $C_{m_1 \ldots m_L}$ to be real.

To proceed, consider the 1-loop dilatation operator acting on the scalar operators (4.1) which is equal (in the large-N limit) to [9]

$$H_{m_1 \cdots m_L, n_1 \ldots n_L} = \frac{\lambda}{(4\pi)^2} \sum_{l=1}^{L} \left(\delta_{m_l m_{l+1}} \delta^{n_l n_{l+1}} + 2\delta_{m_l}^{n_l} \delta_{m_{l+1}}^{n_{l+1}} - 2\delta_{m_l}^{n_{l+1}} \delta_{m_{l+1}}^{n_l} \right) . \tag{4.2}$$

It has a nice interpretation [9] as an integrable Hamiltonian acting on an $SO(6)$ spin chain. Its eigenvalues can be computed using Bethe ansatz techniques [9]. For the purpose of comparison with string theory it is better, however, to use an effective action approach which allows one to describe the low energy states in a universal way. This action can be directly compared with (a limit of) the semiclassical $AdS_5 \times S^5$ string action. Therefore, it is appropriate to say that, in this method, we compare semiclassical solutions on one side with semiclassical solutions on the other.

To obtain this effective action description one can follow [16, 18, 21] and use the method of coherent states (see, e.g., [40]). This method produces a path integral representation for the propagator and therefore one gets a classical action for the system. As an approximation one can just consider classical solutions which is justified since through a rescaling of the coordinates one can see that the inverse of the large momentum acts as the effective Planck constant of the system. A similar idea we discussed in the previous section regarding the commutators (3.28). Instead of following this path we can reach the same result by an alternative method discussed in [18] which is also more suitable for computing corrections to the leading order result (even though we are not going to do this in the present paper).

The first step is to use a factorized ansatz for the operator (4.1) where the matrix $C_{m_1 \ldots m_L}$ is given by the product of L 6-vectors $v_1, \ldots, v_L$

$$\mathcal{O} = \mathrm{tr}\left(\prod_{l=1}^{L} v_{lm} X^m\right) , \quad \text{i.e.} \quad C_{m_1 \ldots m_L} = v_{1m_1} \ldots v_{Lm_L} , \quad \text{or} \quad C = \bigotimes_{l=1}^{L} v_l . \tag{4.3}$$

The case of BPS operators for which C should be totally symmetric and traceless corresponds to all v_l being equal to the same vector v with $v^2 = 0$.

Once more, it proves useful to think of the theory in $R \times S^3$ where we deal with states instead of operators. Each scalar particle can be in 6 possible states $|m\rangle$, $m = 1, \ldots, 6$ (since the fields are real and therefore the antiparticles are the same as the particles). A generic state of a particle is parameterized by six complex numbers up to a normalization condition and an overall irrelevant phase. The operator with coefficient matrix C corresponds to a composite state of L particles each in a state given by $v_m|m\rangle$. The normalization condition implies $vv^* = 1$ and the overall irrelevant phase translates into the gauge invariance $v \to e^{i\beta}v$ as in (3.10). Therefore, the vectors v effectively live in the 10-dimensional space CP^5.

The Schrödinger equation for the wave function of a state associated to C follows from minimizing the action

$$S = -\int dt \left(iC^*_{m_1\ldots m_L} \frac{d}{dt} C_{m_1\ldots m_L} + C^*_{m_1\ldots m_L} H_{m_1\cdots m_L, n_1\ldots n_L} C_{n_1\ldots n_L} \right) . \tag{4.4}$$

In general, variations of C with t may be interpreted as a RG evolution of the "coupling constants" corresponding to the scalar operators (4.1) (with the canonical dimension factor extracted).

Instead of minimizing the action S in the full space of tensors C we shall consider its reduction to the subsector given by the factorized ansatz (4.3). Then we find (suppressing 6-vector indices on L vectors v_l each satisfying $v_l^* v_l = 1$)[18]

$$S = -\int dt \sum_{l=1}^{L} \left\{ iv_l^* \frac{d}{dt} v_l + \frac{\lambda}{(4\pi)^2} \Big[(v_l^* v_{l+1}^*)(v_l v_{l+1}) + 2 - 2(v_l^* v_{l+1})(v_l v_{l+1}^*) \Big] \right\} . \tag{4.5}$$

which is gauge invariant under $v_l \to e^{i\beta_l(t)} v_l$. As expected [9], the Hamiltonian (the term in the square brackets) vanishes for the BPS case when v_l does not depend on l, i.e. $v_l = v$, and $v^2 = 0$. More generally, if we assume that v_l is changing slowly with l (*i.e.* $v_l \simeq v_{l+1}$), then we find that (4.5) contains a potential term $(v_l^* v_l^*)(v_l v_l)$ coming from the first "trace" structure in (4.2). This term will lead to large (order λL [9]) shifts of anomalous dimensions, invalidating a low-energy expansion, i.e. prohibiting one from taking the continuum limit

$$L \to \infty , \qquad \tilde{\lambda} = \frac{\lambda}{L^2} = \text{fixed} , \tag{4.6}$$

[18] The same expression is found by starting with the form of the dilatation operator (4.2) written in terms of the $SO(6)$ generators in the vector representation $M^{ab}_{mn} = \delta^a_m \delta^b_n - \delta^b_m \delta^a_n$ [9] $H = \frac{\lambda}{(4\pi)^2} \sum_{l=1}^{L} H_{l,l+1}$, $H_{l,l+1} = M_l^{mn} M_{l+1}^{mn} - \frac{1}{16}(M_l^{mn} M_{l+1}^{mn})^2 + \frac{9}{4}$ and replacing M_l^{mn} by $< v|M_l^{mn}|v >$ or $v_l^m v_l^{*n} - v_l^n v_l^{*m}$ with $v_l^* v_l = 1$.

and thus from establishing a correspondence with string theory along the lines of [16, 18, 21].[19]

To get the low energy solutions when variations of v_l from site to site are small we are thus to impose the restriction

$$v_l^2 = 0 \ , \qquad\qquad l = 1, \dots, L \tag{4.7}$$

which minimizes the potential energy coming from the first term in the Hamiltonian (4.2). Note that, like $v_l^* v_l = 1$, this condition is to be imposed in a "strong" sense, i.e. with a Lagrange multiplier: one can check that if imposed at $t = 0$ this condition is not preserved by the time evolution implied by the equation following from (4.5). This does not, however, imply a problem of principle for our present goal of comparison with semiclassical states in string theory: mixing between states with $v_l^2 = 0$ and with $v_l^2 \neq 0$ will be suppressed in the continuum limit (4.6) we are interested in (as already mentioned above, states with $v_l^2 \neq 0$ will have large anomalous dimensions in this limit, and the same will apply to off-diagonal elements of the Hamiltonian, i.e. of the anomalous dimension matrix).

The condition (4.7) implies that the operator at *each* site is invariant under half of the supercharges of the $\mathcal{N}$=4 superalgebra. That can be seen from the supersymmetry variation of the operator $v_m X^m$:

$$\delta_\epsilon (v_m X^m) = \frac{i}{2} \bar{\epsilon} v_m \Gamma^m \psi \ , \tag{4.8}$$

where the index m is summed from 1 to 6. If $v^2 = 0$, the matrix $v_m \Gamma^m$ satisfies $(v_m \Gamma^m)^2 = 0$. Then half of its eigenvalues are equal to zero, meaning that the operator $v_m X^m$ is invariant under the supersymmetry variations associated with the null eigenvalues. We may thus call (4.7) a "*local* BPS" condition (see also [16, 27, 28]) since the preserved combinations of supercharges in general are different for each v_l and therefore the complete operator corresponding to $C = \bigotimes v_l$ is not BPS. "Local" should

[19] Note that the same expression (4.5) in the case of real vectors v_l was found in [21] as corresponding to the second $SO(6)/SO(5)$ choice of the $SO(6)$ coherent state with the non-BPS "ground state" $v = (0, 0, 0, 0, 0, 1)$. In this case $v_l^* = v_l$ so that the first (WZ-type) term in (4.5) vanishes and the Lagrangian in (4.5) becomes simply $\frac{\lambda}{(4\pi)^2} \sum_{l=1}^{L} [2 - (v_l v_{l+1})^2]$. Then the potential term is constant and proportional to λL. In this case one cannot consistently define the continuum limit (4.6). This real case is a consistent truncation of the general complex case we consider here provided $\frac{d}{dt} v_l = 0$ (this does not imply the vanishing of the anomalous dimension because v_l are constrained by $v_l^2 = 1$, i.e. there is an additional Lagrange multiplier term in the equation for v_l).

be understood in the sense of the spin chain, or, equivalently, the string world-sheet direction.[20]

As already mentioned above, such states correspond to $SO(6)/[SO(4)\times SO(2)]$ coherent states of $SO(6)$ considered in [21, 28] which are obtained (at each site l) by applying an $SO(6)$ rotation parametrized by an element of the coset $G_{2,6}$ to the vacuum state $v=(0,0,0,0,1,i)$ satisfying $v^2=0$ (note that the condition (4.7) is invariant under $SO(6)$ rotations). The corresponding coherent state action [21] is then indeed equivalent to (4.5) with the corresponding constraints added, i.e.

$$S=-\int dt\sum_{l=1}^{L}\Big\{iv_l^*\frac{d}{dt}v_l+\frac{\lambda}{(4\pi)^2}\left[(v_l^*v_{l+1}^*)(v_lv_{l+1})+2-2(v_l^*v_{l+1})(v_lv_{l+1}^*)\right]$$
$$+\mu_1(v_l^*v_l-1)+\frac{1}{2}(\mu_2v_l^2+c.c)\Big\}\ . \tag{4.9}$$

If we now consider the case when v_l are slowly varying then we can take the continuum limit (4.6) as in [16, 18, 21] by introducing the 2-d field $v_m(t,\sigma)$ with $v_{ml}(t)=v_m(t,\frac{2\pi l}{L})$. Then

$$S=-L\int dt\int_0^{2\pi}\frac{d\sigma}{2\pi}\left\{iv^*\frac{\partial v}{\partial t}+\frac{1}{2}\tilde\lambda(D_1v)(D_1v)^*+\mu_1(v^*v-1)+\frac{1}{2}(\mu_2v^2+c.c)\right\}\ , \tag{4.10}$$

where $\tilde\lambda\equiv\frac{\lambda}{L^2}=\frac{1}{\ell^2}$ and $D_1v=\partial_\sigma v-(v^*\partial_\sigma v)v$ as in (3.11). Note that all higher derivative terms are suppressed in the limit (4.6) by powers of $\frac{1}{L}$ and the potential term is absent due to the condition $v^2=0$. Thus (4.10) becomes equivalent to the $G_{2,6}$ "Landau-Lifshitz" sigma model (3.18) obtained from the string theory action I (3.19) after we rename $v_m\to V_m$ and rescale the time coordinate by $\tilde\lambda^{-1}$ as in (2.25) (and appropriately rescale the Lagrange multipliers). This action has gauge invariance $v_m\to e^{i\beta}v_m$, where β is an arbitrary function of σ and t.

The equations of motion following from (4.10) are

$$i\dot v_m-\frac{1}{2}\tilde\lambda\left[v_m''-2(v^*v')v_m'\right]+\mu_1v_m+\mu_2^*v_m^*=0\ , \tag{4.11}$$

where $v_m^*v_m=1,\ v_m^2=0,\ v^{*2}=0$ and dot and prime stand for derivatives over t and σ. Multiplying this by v_m^* and combing with complex-conjugate equation we learn that $(v_m^*v_m')'=0$, i.e. $v_m^*v_m'=f(t)$, and also find the expression for μ_1. Multiplying (4.11) by v_m we get $\mu_2^*=-\frac{1}{2}\tilde\lambda v_m'v_m'$.

[20] This generalizes the argument implicit in [16]; equivalent proposal was made in [27]. This is related to but different from the "nearly BPS" operators discussed in [25] (which, by definition, were those which become BPS in the limit $\lambda\to 0$).

Besides the equations of motion, we need to add an additional condition. The presence of the trace in (4.1) implies that we have to consider only spin chain states that are invariant under translations in l or in σ. Quantum mechanically, this means that the momentum in the direction σ should vanish: $P_\sigma = 0$. In the low energy effective action description we use here, directly imposing the translation invariance in σ would be too restrictive (this would leave us only with constant fields), but we should still impose the global constraint $P_\sigma = 0$ since the conserved momenta should agree between the semiclassical (coherent) and exact quantum states. From (4.10) we can compute the translational Noether charge P_σ with the result:

$$P_\sigma = \int_0^{2\pi} \frac{d\sigma}{2\pi} T_{01} = L \int_0^{2\pi} \frac{d\sigma}{2\pi}\ i v_m^* \partial_\sigma v_m = 0 \ . \tag{4.12}$$

where $T_{01} = -\frac{\partial \mathcal{L}}{\partial \dot{v}} v'$. This should be viewed as a condition on the solutions $v_m(t,\sigma)$. This condition agrees precisely with (3.16) that was obtained on the string side, after we identify v_m with V_m there.

In the following section we shall find particular solutions of equation (4.11) (and (4.12)) and show that S reproduces the one loop anomalous dimensions obtained previously for rotating and pulsating strings using the Bethe ansatz [10, 15, 31]. We can thus conjecture that, similarly to the previously discussed pure-rotation $SU(2)$ and $SU(3)$ cases, here one has an exact relation between solitons of this "Landau-Lifshitz" sigma model (4.10) and the corresponding set of Bethe ansatz eigenstates, generalizing the one found in the $SU(2)$ case in [17].

Let us make an important comment on the interpretation of the operators associated, according to (4.3), to the solutions of (4.11) and the corresponding string solutions. In the continuum limit we may write the operator (4.3) corresponding to the solution $v(t,\sigma)$ as

$$\mathcal{O} = \mathrm{tr}\Big(\prod_\sigma \mathrm{v}(t,\sigma)\Big) \ , \qquad\qquad \mathrm{v} \equiv v_m(t,\sigma) X^m \ . \tag{4.13}$$

As already mentioned above, such a "coherent" operator representing a coherent state of the spin chain will not in general be an eigen-operator of the dilatation operator or the Hamiltonian (4.2) (in particular, it may have a non-trivial dependence on t). Indeed, as in flat space, where a string state with a given (large) energy can be realized either as a pure Fock eigenstate or as a coherent superposition of Fock space states, here the semiclassical string states represented by classical string solutions should be dual to coherent spin chain states or coherent operators, which are different from the exact eigenstates of the dilatation operator but which should lead to the same energy or anomalous dimension expressions. Once again, the SYM operator naturally associated to a semiclassical string solution should be a locally BPS coherent operator (4.13).

The t-dependence of the string solution thus translates into the RG scale dependence of $\mathcal{O}$, while the σ-dependence describes the ordering of the factors under the trace.[21] At the same time, the Bethe ansatz approach [10, 11, 17] should be determining the exact eigenvalues of the dilatation operator. The reason why the two approaches happen to be in agreement is that in the limit (4.6) we consider the problem is essentially semiclassical, and because of the integrability of the spin chain, its exact eigenvalues are not just well-approximated by the classical solutions but are actually exactly reproduced by them (just as in the harmonic oscillator or flat space string theory case), i.e. the semiclassical coherent state sigma model approach happens to be exact (related observations were made, e.g., in [41]).

5. Particular leading-order solutions

In the previous section we have found an agreement between the actions describing a string moving fast in the S^5 part of $AdS_5 \times S^5$ and the action describing the continuum limit of the spin chain determining the 1-loop anomalous dimensions. In this section we shall consider particular solutions of the equation (4.11)

$$i\dot{v}_m = \frac{1}{2}\left[v_m'' - 2(v^*v')v_m'\right] - \mu_1 v_m - \mu_2^* v_m^* \ , \tag{5.1}$$

which follows from the Lagrangian (3.18) after replacing $V \to v$, or, equivalently, from (4.10) after rescaling time by $\tilde{\lambda}$ (see eq.(4.11)). Such particular solutions include all rotating and pulsating string configurations that have been discussed in the literature and whose motion is only in the S^5.[22] The description of the pulsating solution as a particular solution of the leading-order effective action is new; as a result, we are able to explicitly identify the coherent SYM operators corresponding to pulsating solutions (the form of the Bethe ansatz eigenstates associated to the same energy eigenvalues is implicitly contained in the previous work of [10, 15, 31]).

[21] It should be noted also that not all "long" SYM operators (with large canonical dimension) are represented by the above locally BPS coherent operators. For example, BMN operators (e.g. $\sum c_k$ tr $(X^{J-k}YX^kY)$) are dual to small fluctuations near point-like rotating string which are Fock-space states and which are indeed exact eigenstates of the dilatation operator. Still, their energies can be reproduced by considering linearized solutions of the coherent state LL equation (i.e. by small fluctuations of unit vector $\vec{n}$ near $(0,0,1)$, see [16, 18]). The corresponding operators are then linearizations of the coherent operators corresponding to generic solution of the LL equation. Note also that in contrast to "good" coherent operators dual to classical string states depending on (several) semiclassical parameters, the BMN states depend only on one large quantum number J, and their anomalous dimensions scale as $\frac{\lambda}{L^2}$, $L = J$, and not as $\frac{\lambda}{L}$ as in the case of "true" semiclassical operators.

[22] Pulsating solutions in AdS_5 were considered in [24, 42].

5.1 Rotating strings: $SU(3)$ and $SU(2)$ subsectors

For generic rotating string solutions the agreement between string theory and spin chain at the level of the leading-order effective actions was already demonstrated in [16, 18, 21, 20]. Here we just show how this case is included in our general treatment. To this end we shall use the following ansatz for v in (4.10),(4.13),(5.1)

$$v_m = \frac{1}{\sqrt{2}}(U_1,\, iU_1,\, U_2,\, iU_2,\, U_3,\, iU_3), \quad U_i^* U_i = 1 \;\Rightarrow\; v_m v_m = 0, \quad v_m^* v_m = 1\ . \tag{5.2}$$

This is in agreement with the expectations for a rotating string since v can be written also as

$$v = \frac{1}{\sqrt{2}}\left[U_1\, X + U_2\, Y + U_3\, Z\right]\ , \tag{5.3}$$

where, as before, $X = X_1 + iX_2$, $Y = X_3 + iX_4$, $Z = X_5 + iX_6$, and $(X_k)_m = \delta_{km},\ k = 1, \ldots, 6$ form a basis in R^6. Equivalently, this determines the form of v in (4.13). This ansatz is easily seen to satisfy the equations of motion (5.1) if the complex 3-vector U_i, $i = 1, 2, 3$ satisfies the equations of motion for the Lagrangian

$$\mathcal{L} = -iU_i^* \dot{U}_i - \frac{1}{2}\left[U^{*\prime}_i U_i' + (U_i^* U_i')^2\right] + \Lambda(U_i^* U_i - 1)\ , \tag{5.4}$$

which is just the Lagrangian in (4.10) after the substitution of (5.3). This is the same CP^2 Lagrangian (2.27) for leading-order 3-spin rotating solutions [21] which we rederived (and generalized to all orders) in section 2. The $SU(2)$ case of CP^1 model describing 2-spin solutions [16] is obtained by setting $U_3 = 0$. In that case a convenient parametrization is

$$U_1 = \sin\frac{\theta}{2}\ e^{\frac{i}{2}(\phi-\varphi)}, \qquad U_2 = \cos\frac{\theta}{2}\ e^{\frac{i}{2}(\phi+\varphi)}\ . \tag{5.5}$$

If we write the Lagrangian for these fields, ϕ disappears due to gauge invariance. The Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}\cos\theta\ \dot{\varphi} - \frac{1}{8}(\theta'^2 + \sin^2\theta\ \varphi'^2)\ . \tag{5.6}$$

It is an interesting exercise to evaluate the next correction by substituting this ansatz into (3.39) and thus reproduce the result of [18].

5.2 Pulsating strings: $SO(3)$ subsector

To analyze the case of pulsating strings let us first discuss which operators should be dual to them. In the case [24] where the string is pulsating (with no rotation) we can

consider the motion to be restricted to $S^2 \subset S^5$, i.e. to occur in the coordinates X_1, X_2, X_3, with $X_1^2 + X_2^2 + X_3^2 = 1$. Using the standard polar coordinates (θ, ϕ) on the sphere, the solution is of the form $\theta = \theta(\tau)$, $\phi = m\sigma$, with m being an integer. While the exact form of the function $\theta(\tau)$ is obtained in section 6, here we will only need to know that it is approximately given by $\theta \simeq t$, so that θ plays the role of the fast coordinate. Namely, each piece (or bit) of the closed pulsating string moves along a maximal radius circle of constant longitude ϕ at almost the speed of light. From the generic correspondence we discussed in the previous sections we expect that each piece of the string maps into the scalar operator (4.13) with v of the following form

$$\mathrm{v}(\sigma) = \frac{1}{\sqrt{2}} \left[\cos\phi \ X_1 + \sin\phi \ X_2 + iX_3\right] \ , \qquad \phi = m\sigma \ , \tag{5.7}$$

i.e. the corresponding 6-vector is $v_m = \frac{1}{\sqrt{2}}(\cos\phi, \sin\phi, i, 0, 0, 0)$ which satisfies $v^* v = 1, \ v^2 = 0$.

We can check that this is a solution of the reduced action by considering the special case of (3.18) or, equivalently, (4.10) when the motion occurs only in 3 out of 6 directions, i.e. within S^2. In general, writing (4.10) in terms of 6 real coordinates

$$v_m \equiv \frac{1}{\sqrt{2}}(a_m + i b_m) \ , \qquad a_m^2 = 1 \ , \quad b_m^2 = 1 \ , \quad a_m b_m = 0 \ , \tag{5.8}$$

we get (renaming the Lagrange multipliers)

$$\mathcal{L} = a_m \dot{b}_m - \frac{1}{4}\Big[a_m'^2 + b_m'^2 - 2(a_m b_m')^2\Big] + \Lambda(a_m^2 - 1) + \tilde{\Lambda}(b_m^2 - 1) + \ \mu \ a_m b_m \ . \tag{5.9}$$

This Lagrangian still has gauge invariance associated with an arbitrary phase transformation of v, allowing one to impose one real gauge condition. That leaves us with $2 \times 6 - 3 - 1 = 8$ real independent functions.

Let us now restrict to the case when m runs only 1,2,3, i.e. the corresponding string motions happen in S^2 part of S^5. Then we should have $2 \times 3 - 3 - 1 = 2$ independent real variables. We can fix the gauge invariance by the condition $b_3 = 0$; then the unit 3-vector $b = (b_1, b_2, 0)$ orthogonal to the unit vector $a = (\sin\vartheta \ \cos\phi, \sin\vartheta \ \sin\phi, \cos\vartheta)$ can be chosen as $b = (\sin\phi, -\cos\phi, 0)$. Then (5.9) is found to be

$$\mathcal{L} = \sin\vartheta \ \dot{\phi} - \frac{1}{4}(\vartheta'^2 + \cos^2\vartheta \ \phi'^2) \ , \tag{5.10}$$

which is formally equivalent to the LL sigma model [16, 18] found in the $SU(2)$ rotating sector where strings are moving in S^3 part of S^5. There, one of the three angles of S^3 was a fast variable, and eliminating it, we were left with two dynamical variables. Here,

we have two coordinates of S^2 to start with; eliminating one fast coordinate we are left with a "slow" coordinate, but we are still to keep *one more* degree of freedom which may be interpreted as a momentum conjugate to the "slow" coordinate.

A special solution is $\vartheta = \frac{\pi}{2},\ \phi = m\sigma$ which corresponds to the pulsating solution we discussed above. Taking into account that, with the Lagrangian (5.10), the action is given by (4.10), i.e.

$$S = L \int dt \int \frac{d\sigma}{2\pi}\ \mathcal{L} \tag{5.11}$$

where t is related to the time in (5.10) by the rescaling (2.25) with $\tilde{\lambda} = \frac{\lambda}{L^2}$, we obtain the energy of the pulsating solution as

$$E = \frac{1}{4}\tilde{\lambda} L\, \cos^2\vartheta\ \phi'^2 = \frac{\lambda}{4L} m^2\ , \tag{5.12}$$

which is in agreement with the result of [24] (see also eq.(6.22) for $J = 0$.)

5.3 Pulsating and rotating strings: $SO(4)$ subsector

The case of a pulsating *and* rotating string [15] is slightly more involved. Here the motion occurs in $S^3 \subset S^5$ parametrized by $X_1, \ldots, X_4,\ X^2 = 1$. In the standard parametrization

$$X_1 = \sin\theta\cos\phi_1, \quad X_2 = \sin\theta\sin\phi_1, \quad X_3 = \cos\theta\cos\phi_2, \quad X_4 = \cos\theta\sin\phi_2, \tag{5.13}$$

the metric on S^3 is

$$ds^2 = d\theta^2 + \sin^2\theta\ d\phi_1^2 + \cos^2\theta\ d\phi_2^2\ , \tag{5.14}$$

and the string ansatz is then

$$\theta = \theta(\tau)\ , \quad \phi_1 = \phi_1(\tau), \quad \phi_2 = m\sigma\ . \tag{5.15}$$

The closed string moves approximately at the speed of light along a tilted maximal circle in the sphere S^2 parametrized by (θ, ϕ_1). Let us denote the angle between the normal to the circle and the $\theta = \frac{\pi}{2}$ plane as γ. In terms of conserved quantities it is given by

$$\sin\gamma = \frac{J}{L}\ , \tag{5.16}$$

where J is the momentum conjugate to ϕ_1 and L is defined through $L = J + I_\theta$, where I_θ is the action integral corresponding to the coordinate θ (see next section for more details).

Using the same arguments as above, namely associating each point of the string moving along a maximum circle with the corresponding operator carrying the same

R-charge, we can identify this string configuration with an operator as in (4.13) where v is given by:

$$\mathrm{v}_{\text{leading order}} = \frac{1}{\sqrt{2}}\left[iX_1 + \sin\gamma\ X_2 + \cos\gamma\ \cos\phi_2\ X_3 + \cos\gamma\ \sin\phi_2\ X_4\right] , \tag{5.17}$$

where we have emphasized that this identification is valid at the leading order which is equivalent to considering ϕ_2 as being constant (*i.e.* $m = 0$ in (5.15)). For $\phi_2 = m\sigma$, $m \neq 0$, the 6-vector v_m corresponding to (5.17) is not a solution of the equation (5.1). To get a consistent solution we use eq.(5.17) as a guide and propose to look for v_m as

$$v = v_0(t) + w(\sigma) , \tag{5.18}$$

with v_0 being a σ-independent complex vector having components along X_1 and X_2 only. The vector w is real and has components along X_3, X_4. The constraints $v^2 = 0$ and $vv^* = 1$ imply

$$1 - v_0^* v_0 = w^2 = -v_0^2 , \tag{5.19}$$

so that, in particular, v_0^2 and $|v_0|^2$ should be time-independent. Taking this into account, and using as a further assumption that the Lagrange multipliers μ_1 and μ_2 are real constants we find that (5.1) reduces to

$$i\dot{v}_0 = -\mu_1 v_0 - \mu_2 v_0^* , \qquad 0 = w'' - 2(\mu_1 + \mu_2) w . \tag{5.20}$$

The equation for v_0 is easily solved. We get

$$v_0 = Ae^{i\omega t} + Be^{-i\omega t} , \tag{5.21}$$

with $\omega = \sqrt{\mu_1^2 - \mu_2^2}$ and A, B are complex two component vectors that have to satisfy

$$A^2 = 0 , \qquad B^* = -\frac{\mu_2}{\mu_1 + \omega} A , \qquad |A|^2 = \frac{(\mu_1 + \omega)^2}{(\mu_1 + \omega + \mu_2)^2} . \tag{5.22}$$

For the other vector w we can choose the expression suggested by (5.17), i.e. $w = \frac{1}{\sqrt{2}}(0, 0, \cos\gamma\cos m\sigma, \cos\gamma\sin m\sigma)$ which implies $\mu_1 + \mu_2 = -\frac{1}{2}m^2$ and $w^2 = \frac{1}{2}\cos^2\gamma$. Since $v_0^2 = -w^2$ we can use (5.22) to find that

$$\mu_1 = -\frac{m^2}{4}(1 + \sin^2\gamma) , \qquad \mu_2 = -\frac{m^2}{4}\cos^2\gamma , \qquad \omega = -\frac{m^2}{2}\sin\gamma . \tag{5.23}$$

Choosing an arbitrary phase to obtain an operator similar to (5.17) we can then write the solution as

$$v = \frac{1}{\sqrt{2}}(-\sin\gamma\sin\omega t + i\cos\omega t,\ \ \sin\gamma\cos\omega t + i\sin\omega t,\ \cos\alpha\cos m\sigma,\ \ \cos\gamma\sin m\sigma) . \tag{5.24}$$

For this solution we can compute the angular momenta $M_{mn} = \int_0^{2\pi} \frac{d\sigma}{2\pi} \mathcal{M}_{mn}$ using (3.21). The only non-vanishing densities are:

$$\mathcal{M}_{12}(\sigma) = L \sin\gamma \ , \quad \mathcal{M}_{13}(\sigma) + i\mathcal{M}_{23}(\sigma) = L\ e^{i\omega t} \cos m\sigma \cos\gamma \ , \tag{5.25}$$

$$\mathcal{M}_{14}(\sigma) + i\mathcal{M}_{24}(\sigma) = L\ e^{i\omega t} \sin m\sigma \cos\gamma \ . \tag{5.26}$$

Integrating over σ we find that the only non-vanishing angular momentum component is

$$M_{12} = \int \frac{d\sigma}{2\pi} \mathcal{M}_{12}(\sigma) = L \sin\gamma = J \ , \tag{5.27}$$

in agreement with the definition $\sin\gamma = \frac{J}{L}$ in (5.16). While it was clear that J is the only non-vanishing angular momentum, computing the densities $\mathcal{M}_{mn}(\sigma)$ is instructive since the result agrees with the expectations from the full string solution.

The energy can be evaluated from (4.10) using that

$$v^* v' = 0 \ , \qquad v' v'^* = \frac{1}{2} m^2 \cos\gamma \ , \tag{5.28}$$

with the result

$$E = \frac{1}{4} \tilde{\lambda}\, L\, m^2 \cos^2\gamma = \frac{\lambda m^2}{4L} \left(1 - \frac{J^2}{L^2}\right) \ , \tag{5.29}$$

which is in agreement with [15] (see also eq.(6.22)). It is also instructive to substitute this solution into (3.39) and obtain the next correction to the energy which can be seen to agree with the expansion of the exact energy expression in (6.22).

Interpreted on the spin chain side using (4.13), the solution (5.24) for $\mathrm{v} = v_m X^m$ thus determines the structure of the coherent SYM operator which is dual to the pulsating and rotating string solution.

6. Exact solution for pulsating and rotating string in $SO(4)$ sector

In this section we shall study the full classical solution corresponding to a string rotating and pulsating in the S^3 part of S^5. This solution was discussed in [15]. Nevertheless, we shall describe it here using a different (and straightforward) approach which appears to have some technical and conceptual advantages. One important point is that the use of action–angle variables will allow us to obtain closed expressions for the relation between the energy and the conserved momenta. These expressions can be easily expanded at large energy to obtain the energy as a power series in the momenta and winding number, much in the same way as was done for the case of the two-spin rotating string in [7, 32, 11].

Furthermore, we will see that, in the limit of large energy, the rotation is along a maximal circle precessing around a fixed axis, which turns out to be precisely the motion described by the reduced sigma model of the previous section.

Before starting let us point out that the solution we are going to discuss is a special case of a more general class of pulsating and rotating solutions where the string may be stretched and rotating in all the three planes of S^5 [13, 8]. The corresponding integrable Neumann model is "2-d dual" ($\tau \leftrightarrow \sigma$) to the rotating string model [13, 8]. Let us choose $t = \kappa\tau$ together with the following ansatz ($i = 1, 2, 3$) [13]

$$X_i = z_i(\tau)\ e^{im_i\sigma}\ , \qquad z_i = r_i(\tau)\ e^{i\alpha_i(\tau)}\ , \qquad \sum_{i=1}^{3} r_i^2(\tau) = 1\ , \tag{6.1}$$

where the "winding numbers" m_i must take integer values in order to satisfy the closed string periodicity condition. This ansatz describes a pulsating and rotating string in S^5 (special cases were discussed previously in [33, 5, 24, 43, 15]). The corresponding 1-d effective Lagrangian is

$$\mathcal{L} = \frac{1}{2}\sum_{i=1}^{3}(\dot{z}_i\dot{z}_i^* - m_i^2 z_i z_i^*) + \frac{1}{2}\Lambda(\sum_{i=1}^{3} z_i z_i^* - 1)\,. \tag{6.2}$$

Solving for $\dot{\alpha}_i$ we get $r_i^2\dot{\alpha}_i = \mathcal{J}_i$=const, where $\mathcal{J}_i$ are the angular momenta. Then we end up with

$$\mathcal{L} = \frac{1}{2}\sum_{i=1}^{3}(\dot{r}_i^2 - m_i^2 r_i^2 - \frac{\mathcal{J}_i^2}{r_i^2}) + \frac{1}{2}\Lambda(\sum_{i=1}^{3} r_i^2 - 1)\,. \tag{6.3}$$

Thus pulsating solutions carrying 3 spins $\mathcal{J}_i$ are described by a special Neumann-Rosochatius integrable system [13]. Since the corresponding conformal gauge constraints are also $\tau \leftrightarrow \sigma$ symmetric, they take a form similar to the one in pure-rotation case

$$\kappa^2 = \sum_{i=1}^{3}(\dot{r}_i^2 + m_i^2 r_i^2 + \frac{\mathcal{J}_i^2}{r_i^2})\ , \qquad \sum_{i=1}^{3} m_i\mathcal{J}_i = 0\ . \tag{6.4}$$

One is thus to look for periodic solutions of (6.3) subject to (6.4), i.e. having finite 1-d energy. The general real 6-d Neumann system has the following six commuting integrals of motion ($z_i = x_i + ix_{i+3}$):

$$F_m = x_m^2 + \sum_{m\neq n}^{6}\frac{(x_m x_n' - x_n x_m')^2}{w_m^2 - w_n^2}\,, \qquad \sum_{m=1}^{6} F_m = 1\ , \tag{6.5}$$

but in the present case when 3 of the 6 frequencies are equal ($w_i = w_{i+3} = m_i$) one needs to consider the 3 non-singular combinations of F_m which then give the 3 integrals of (6.2): $I_i = F_i + F_{i+3}$, or, explicitly,

$$I_i = r_i^2 + \sum_{j\neq i}^{3} \frac{1}{m_i^2 - m_j^2}\Big[(r_i r_j' - r_j r_i')^2 + \frac{\mathcal{J}_i^2}{r_i^2} r_j^2 + \frac{\mathcal{J}_j^2}{r_j^2} r_i^2\Big] \ . \tag{6.6}$$

This gives two additional (besides $\mathcal{J}_i$) independent integrals of motion, explaining why this system is completely integrable.

6.1 Form of the solution

In the simplest "elliptic" special case we shall consider here the string is stretched in 34 plane, while rotating in 12 plane, i.e. $m_2 = m,\ m_1 = m_3 = 0$ and $\mathcal{J}_1 = \mathcal{J},\ \mathcal{J}_2 = \mathcal{J}_3 = 0$. This choice corresponds to a particle on S^2 with angular momentum $\mathcal{J}$ and an oscillator potential in the third direction. Equivalently, in terms of 6 real coordinates this corresponds to the following ansatz: $X_1 = x_1(\tau),\ X_2 = x_2(\tau),\ X_3 + iX_4 = x_3(\tau)e^{im\sigma},\ X_{5,6} = 0$, where $x_1 = \sin\theta\ \cos\phi_1,\ x_2 = \sin\theta\ \sin\phi_1,\ x_3 = \cos\theta$. Explicitly, the conformal gauge action for the string moving in $R_t \times S^3$ with the S^3 metric given in the angular parametrization by (5.14) is

$$I = -\frac{1}{2}\sqrt{\lambda}\int d\tau \int_0^{2\pi} \frac{d\sigma}{2\pi}\left[-(\partial_p t)^2 + (\partial_p \theta)^2 + \sin^2\theta\,(\partial_p\phi_1)^2 + \cos^2\theta\,(\partial_p\phi_2)^2\right] \ , \tag{6.7}$$

and the resulting equations of motion are satisfied by the ansatz

$$t = \kappa\tau \ , \quad \theta = \theta(\tau) \ , \quad \phi_1 = \phi_1(\tau) \ , \quad \phi_2 = m\sigma \ , \tag{6.8}$$

provided the functions $\theta(\tau)$ and $\phi_1(\tau)$ satisfy

$$\partial_\tau\left(\sin^2\theta\ \dot\phi_1\right) = 0 \ , \tag{6.9}$$

$$\ddot\theta + (m^2 + \dot\phi_1^2)\cos\theta\sin\theta = 0 \ . \tag{6.10}$$

These equations are equivalent to the conservation of the angular momentum J and the energy E

$$J = \sqrt{\lambda}\ P_{\phi_1} = \sqrt{\lambda}\mathcal{J} \ , \qquad\qquad E = \sqrt{\lambda}\ P_t = \sqrt{\lambda}\ \kappa \ . \tag{6.11}$$

The first equation implies

$$\dot\phi_1 = \frac{\mathcal{J}}{\sin^2\theta} \ . \tag{6.12}$$

The second can then be integrated once becoming the conformal gauge constraint

$$\dot{\theta}^2 + \frac{\mathcal{J}^2}{\sin^2\theta} + m^2\cos^2\theta = \kappa^2 \ . \tag{6.13}$$

As already mentioned above, this equation can be interpreted as describing a periodic motion of a particle of unit mass on S^2 under the influence of an oscillator potential $\mathcal{U}(\theta) = \frac{1}{2}m^2 x_3^2 = \frac{1}{2}m^2\cos^2\theta$.

Eq. (6.13) can be integrated to obtain $\theta(t)$ as

$$\cos\theta = -a_- \ \mathbf{sn}(\frac{ma_+}{\kappa}t) \ , \tag{6.14}$$

where $\mathbf{sn}$ denotes one of Jacobi's elliptic functions with modulus $k = a_-/a_+$, where $a_\pm$ are constants of the motion given by

$$a_\pm^2 = \frac{\kappa^2 + m^2 \pm \sqrt{(\kappa^2+m^2)^2 - 4m^2(\kappa^2 - \mathcal{J}^2)}}{2m^2} \ . \tag{6.15}$$

Notice that $a_\pm$ are real since $\kappa^2 \geq \mathcal{J}^2$ as can be seen from (6.13) by considering the special point $\theta = \pi/2$. The motion of ϕ_1 can be obtained by integrating $d\phi_1/d\theta$, which gives

$$\phi_1 = -\frac{\mathcal{J}}{ma_+}\mathbf{\Pi}\left(\arcsin(\frac{1}{a_-}\cos\theta(t)), a_-^2, \frac{a_-}{a_+}\right) \ , \tag{6.16}$$

where $\mathbf{\Pi}$ is a standard elliptic integral:[23]

$$\mathbf{\Pi}(\psi, n, k) = \int_0^\psi \frac{d\beta}{(1 - n\sin^2\beta)\sqrt{1 - k^2\sin^2\beta}} \ . \tag{6.17}$$

As we can see, the motion on the sphere is separable in the variables θ, ϕ_1. The system has two independent conserved momenta (being a special case of the Neumann system, see above): one is J (which is the same as the action variable for ϕ_1, $I_{\phi_1} = \sqrt{\lambda}\int \frac{d\phi_1}{2\pi}P_{\phi_1} = J$), and the other one we can choose to be the action variable corresponding to θ:

$$\begin{aligned} I_\theta &= \sqrt{\lambda}\oint \frac{d\theta}{2\pi}P_\theta = \sqrt{\lambda}\oint\frac{d\theta}{2\pi}\sqrt{\kappa^2 - \frac{\mathcal{J}^2}{\sin^2\theta} - m^2\cos^2\theta} \\ &= \frac{4\sqrt{\lambda}}{2\pi}\frac{m}{a_+}\left[(a_-^2 - 1)\mathbf{K}\left(\frac{a_-}{a_+}\right) + a_+^2\mathbf{E}\left(\frac{a_-}{a_+}\right) + (a_+^2-1)(a_-^2-1)\mathbf{\Pi}\left(\frac{\pi}{2}, a_-^2, \frac{a_-}{a_+}\right)\right] \ , \end{aligned} \tag{6.18}$$

where $\mathbf{K}$, $\mathbf{E}$ and $\mathbf{\Pi}$ are again the standard elliptic integrals and the integration is over a complete period of motion which takes places between a minimum and a maximum

[23] We follow the notation of [47].

values of θ. These can be found from the zeros of the square root under the integral. The functions $a_\pm$ are defined in eq.(6.15).

An interesting special case is when $J = 0$ which is the pulsating string of [24]. For $J = 0$ eq. (6.15) implies $a_+ = \kappa/m$ and $a_- = 1$. Then (6.18) simplifies to

$$\frac{L}{E} = \frac{I_\theta}{E} = \frac{2}{\pi}\,\mathbf{E}\left(\frac{\sqrt{\lambda}m}{E}\right), \tag{6.19}$$

where we used that $E = \sqrt{\lambda}\kappa$ and identified L with I_θ since I_θ is the only remaining conserved quantity.

6.2 Expansion of the energy

To compare with string theory we are interested in the limit of large energy, namely large $\kappa = \frac{E}{\sqrt{\lambda}}$. The expression (6.18) for the action variable I_θ can be expanded in this limit with the result

$$I_\theta = E - J - \frac{\lambda m^2}{4E}\left(1 - \frac{J^2}{E^2}\right) + \dots\ . \tag{6.20}$$

It is natural to introduce another conserved quantity which is the sum of I_θ and J

$$L \equiv I_\theta + J\ . \tag{6.21}$$

To leading order of the large energy expansion $L \equiv \sqrt{\lambda}\ell$ is equal to the energy and thus $\ell \to \infty$ can be identified with an analogous expansion parameter in [15]. Inverting this expansion, we obtain $E(J, L, \lambda)$ as

$$\begin{aligned} E = L + \frac{\lambda m^2}{4\,L}\left(1 - \frac{J^2}{L^2}\right)\Bigg[1 &- \frac{\lambda m^2}{16\,L^2}\left(1 + 3\frac{J^2}{L^2}\right) + \frac{\lambda^2 m^4}{64\,L^4}\ \left(1 - 3\frac{J^2}{L^2}\right)\left(1 - 5\frac{J^2}{L^2}\right) \\ &- \frac{\lambda^3 m^6}{4096\,L^6}\left(13 + 113\frac{J^2}{L^2} - 1017\frac{J^4}{L^4} + 1211\frac{J^6}{L^6}\right) + \dots\Bigg]\ . \end{aligned} \tag{6.22}$$

Here we computed several higher-order terms in the expansion which can be easily done with a computer algebra program.[24] The energy or, more precisely, the "energy density" E/L thus has an analytic expansion in $\tilde{\lambda} = \frac{\lambda}{L^2}$ and $\frac{J}{L}$. The special case of $L = J$ when $E = J$ corresponds to a degenerate BPS point when the solution becomes a point-like geodesic: in this case $I_\theta = 0$, $\dot{\theta} = 0$, and then (6.10) implies that $\theta = \frac{\pi}{2}$, in which case the length of the string vanishes. Another case of interest is $J = 0$ which is the purely pulsating string of [24].

[24]While this paper was being written there appeared ref. [31] which also computed the first three leading terms of the expansion of E/L using a different (and apparently more complicated) method. Our results agree.

6.3 Relation to leading-order solution

It is instructive to analyze the motion of the string in the large energy limit. On the sphere (θ, ϕ_1) the string is seen as a point-like particle moving almost at the speed of light along a maximal circle. The $m^2 \cos^2\theta$ potential produces an attractive force towards the equator. In each cycle the average force is zero but there is a torque which makes the averaged angular momentum precess around the z axis as depicted in figure 1. This precession is described by the leading order solution. The parameters

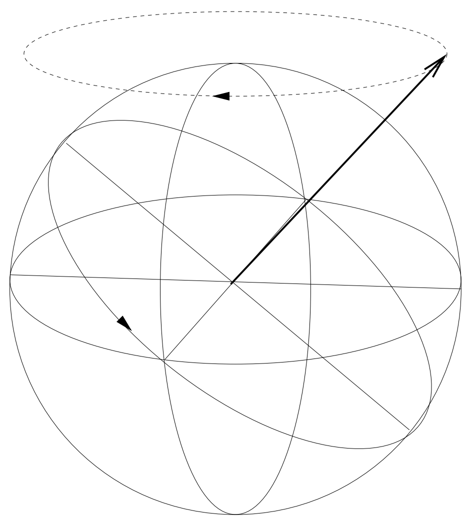

Figure 1: For large angular momentum the effective particle moves on the two-sphere parameterized by (θ, ϕ_1) following a circle almost at the speed of light. The potential produces a force towards the equator which on average translates into a torque perpendicular to the angular momentum, causing a slow precession of the plane of rotation.

characterizing the motion are the period of the motion and the rate of precession. The period T, or, equivalently, the angular frequency $w = 2\pi/T$ can be computed from eq.(6.14) using the fact that the elliptic function $\mathbf{sn}(u)$ with modulus k is periodic with period $2\mathbf{K}(k)$. The result is:

$$\omega = \frac{2\pi}{T} = \frac{\pi m a_+}{2\kappa \mathbf{K}\left(\frac{a_-}{a_+}\right)} \simeq 1 - \frac{\lambda m^2}{4L^2}\left(1 - 3\frac{J^2}{L^2}\right) + \frac{\lambda^2 m^4}{64L^4}\left(3 + 10\frac{J^2}{L^2} - 21\frac{J^4}{L^4}\right) + \ldots, \tag{6.23}$$

where we expanded in large energy κ and used eq.(6.22) to write the result as a function of L and J. Alternatively, we can use the fact that we have already expressed the energy

as a function of the action variables to compute the frequency as

$$\omega = \frac{\partial E}{\partial I_\theta} = \frac{\partial E}{\partial L} \ . \tag{6.24}$$

Using (6.22) we then reobtain (6.23) in a simpler way.

The rate of precession can be found by expanding (6.16) for large κ and computing the variation of ϕ_1 in one cycle of θ:

$$\Delta\phi_1 = \mathcal{J}\frac{4}{ma_+}\mathbf{\Pi}\left(\frac{\pi}{2}, a_-^2, \frac{a_-}{a_+}\right) \simeq 2\pi\left[1 - \frac{\lambda m^2 J}{2L^3} - \frac{3\lambda^2 m^4 J}{16L^5}\left(1 - 3\frac{J^2}{L^2}\right) + \ldots\right] \tag{6.25}$$

We see that there is a precession, since, in one cycle, ϕ_1 advances slightly less than 2π. The rate of precession is therefore

$$\frac{1}{2\pi}\left(\Delta\phi_1 - 2\pi\right) = -\lambda\frac{m^2 J}{2L^3} - \frac{3\lambda^2 m^4 J}{16L^5}\left(1 - 3\frac{J^2}{L^2}\right) + \ldots \tag{6.26}$$

The sign here is in agreement with the direction of the torque (see fig. 1). The above expression can also be computed directly from eq.(6.22) using[25]

$$\Delta\phi_1 = \omega_{\phi_1} T = 2\pi\frac{\omega_{\phi_1}}{\omega} = 2\pi\frac{\left(\frac{\partial E}{\partial J}\right)_{I_\theta}}{\left(\frac{\partial E}{\partial L}\right)_J} = 2\pi\left[1 + \frac{\left(\frac{\partial E}{\partial J}\right)_L}{\left(\frac{\partial E}{\partial L}\right)_J}\right] \ . \tag{6.27}$$

Using again (6.22) we reproduce (6.26).

We conclude that, thanks to the identification of J and L with the action variables, the function $E(J, L)$ can be used to directly compute all the characteristic frequencies of the motion.

At leading order, these frequencies can be also obtained from the reduced sigma model (4.10) and, therefore, on the spin chain side, describe the evolution of the corresponding coherent state.

Having computed the relevant frequencies we can expand (6.14) at large energy or small $\tilde\lambda$ to find that

$$\cos\theta \simeq -\sqrt{1 - \frac{J^2}{L^2}}\left\{\sin\omega t + \frac{\lambda m^2}{16L^2}\left[\left(1 - 5\frac{J^2}{L^2}\right)\sin\omega t + \left(1 - \frac{J^2}{L^2}\right)\sin 3\omega t\right] + \ldots\right\} \ . \tag{6.28}$$

We can see here the typical properties of a perturbative expansion in classical mechanics. Higher harmonics of the fundamental frequency ω appear in the higher order

[25] This relation is valid even if ϕ_1 is not an angle variable, see next subsection for a more precise derivation.

corrections. At the same time, the fundamental frequency itself has a perturbative expansion which we obtained in eq.(6.23). Notice that attempting to expand (6.28) in powers of $\tilde{\lambda} = \frac{\lambda}{L^2}$ by first plugging in (6.23) would result in an incorrect conclusion that there are perturbative terms linearly growing in time. These are called secular or resonant terms and should be eliminated, as was done here by correcting the fundamental frequency. In the canonical formalism this is achieved by adding the average value (over a cycle) of the perturbative Hamiltonian. This gives the corrected form of $E(J,L)$ which, as we saw, determines the corrected frequencies. Furthermore, the angle-dependent part of the Hamiltonian is then eliminated by means of a canonical transformation. If we want to describe the system in terms of the original variables, undoing this canonical transformation produces terms with higher harmonics of the fundamental frequency.

6.4 Canonical perturbation

In the previous subsection we solved exactly the equation of motion for the pulsating solutions and expanded them for large $\ell = \frac{L}{\sqrt{\lambda}}$ (with $\frac{J}{L}$ fixed). If one is interested just in the large ℓ expansion it is possible to avoid the computation of the exact solution. Assuming that the only σ dependence is in $\phi_2(\sigma) = m\sigma$, and using conservation of J to eliminate the variable ϕ_1, we reduce the problem to the study of the Hamiltonian

$$H = \frac{1}{2}P_\theta^2 + \frac{1}{2}\frac{\mathcal{J}^2}{\sin^2\theta} + \frac{1}{2}m^2\cos^2\theta \ . \tag{6.29}$$

For large ℓ, the (bounded) term proportional to m^2 can be consider as a perturbation and we can use the methods described in Appendix, which, for the first order corrections, are actually simpler than the expansion of the previous subsection. At lowest order, i.e. ignoring the term proportional to m^2, energy conservation implies

$$P_\theta = \sqrt{2H_0 - \frac{\mathcal{J}^2}{\sin^2\theta}} \ , \tag{6.30}$$

where H_0 is the value of H at $m = 0$. This determines the action variable to be

$$I_\theta = \sqrt{\lambda}\oint \frac{d\theta}{2\pi}\sqrt{2H_0 - \frac{\mathcal{J}^2}{\sin^2\theta}} = \sqrt{2\lambda H_0} - J \ . \tag{6.31}$$

The canonical transformation from (θ, P_θ) to action–angle variables $(\bar{\theta}, I_\theta)$ is generated by a function $S(\theta, I_\theta)$ such that $\bar{\theta} = \frac{\partial S}{\partial I_\theta}$, and $P_\theta = \frac{\partial S}{\partial \theta}$. Integrating the latter equation and then differentiating the result with respect to I_θ we get

$$\bar{\theta} = \frac{\partial S}{\partial I_\theta} = \int_{\frac{\pi}{2}}^{\theta} d\theta \frac{I_\theta + J}{\sqrt{(I_\theta + J)^2 - \frac{J^2}{\sin^2\theta}}} = \arcsin\left(\frac{L\cos\theta}{\sqrt{L^2 - J^2}}\right) \ , \tag{6.32}$$

where we introduced

$$L = I_\theta + J = \sqrt{\lambda}\ \ell\ , \qquad \ell = \sqrt{2H_0}\ .$$

Equivalently, we can write (6.32) as

$$\cos\theta = \sqrt{1 - \frac{J^2}{L^2}}\sin\bar{\theta}\ . \tag{6.33}$$

Therefore, the Hamiltonian becomes

$$H = \frac{1}{2}\ell^2 + \frac{1}{4}m^2\left(1 - \frac{J^2}{L^2}\right) - \frac{1}{4}m^2\left(1 - \frac{J^2}{L^2}\right)\cos 2\bar{\theta}\ . \tag{6.34}$$

The small parameter is $\frac{m}{\ell} = \frac{m\sqrt{\lambda}}{L}$. The canonical method reviewed in Appendix proceeds by choosing a canonical transformation so that to eliminate all $\bar{\theta}$ dependence in the transformed Hamiltonian. It is convenient to separate the perturbation H_1 into average $\langle H_1\rangle$ and fluctuating $\{H_1\}$ parts as follows:

$$H_1 = \langle H_1\rangle + \{H_1\} = \frac{1}{4}m^2\left(1 - \frac{J^2}{L^2}\right) - \frac{1}{4}m^2\left(1 - \frac{J^2}{L^2}\right)\cos 2\bar{\theta}\ , \tag{6.35}$$

$$\langle H_1\rangle = \frac{1}{4}m^2\left(1 - \frac{J^2}{L^2}\right)\ , \tag{6.36}$$

$$\{H_1\} = -\frac{1}{4}m^2\left(1 - \frac{J^2}{L^2}\right)\cos 2\bar{\theta}\ . \tag{6.37}$$

To second order the transformed Hamiltonian is

$$\bar{H} = H_0 + \langle H_1\rangle + \frac{1}{2}\langle\ [W_1, \{H_1\}]\ \rangle\ , \tag{6.38}$$

where W_1 is obtained from

$$[W_1, H_0] = \frac{\partial H_0}{\partial J}\frac{\partial W_1}{\partial\theta} = -\{H_1\}\ . \tag{6.39}$$

This gives

$$W_1 = \frac{m^2}{8\ell}\left(1 - \frac{J^2}{L^2}\right)\sin 2\bar{\theta}\ . \tag{6.40}$$

For H we obtain

$$H = \frac{1}{2}\frac{L^2}{\lambda} + \frac{1}{4}\left(1 - \frac{J^2}{L^2}\right) - \frac{m^4}{64}\frac{\lambda^2}{L^4}\left(1 - \frac{J^2}{L^2}\right)(5J^2 - L^2)\ , \tag{6.41}$$

which is in agreement with eq.(6.22) as can be seen by noting that $H = \frac{1}{2}\kappa^2 = \frac{1}{2}\frac{E^2}{\lambda}$.

We can use the canonical transformation to obtain $\theta(\tau)$ as in (6.28). The first transformation is (6.33). Then we have to use the transformation generated by W_1. At first order this gives

$$\begin{aligned}\cos\theta &\simeq \sqrt{1-\frac{J^2}{L^2}}\sin\bar\theta + [W_1, \sqrt{1-\frac{J^2}{L^2}}\sin\bar\theta] \\ &= \sqrt{1-\frac{J^2}{L^2}}\left\{\sin\bar\theta + \frac{\lambda m^2}{16L^2}\left[\left(1-5\frac{J^2}{L^2}\right)\sin\bar\theta + \left(1-\frac{J^2}{L^2}\right)\sin 3\bar\theta\right]\right\} \ , \qquad (6.42)\end{aligned}$$

where we used the value of W_1 given by (6.40). If we remember that $\bar\theta = \frac{\partial E}{\partial L}t$ then we see that we get precisely (6.28).

We can also use this method to compute the time dependence of the angle ϕ_1. The first point to notice is that although J is an action variable, ϕ_1 is not the corresponding angle variable as can easily be seen since $\dot\phi_1$ is not constant. The angle variable can be obtained by solving the Hamilton-Jacobi equation which gives the action

$$S = J\phi_1 + \int d\theta\sqrt{L^2 - \frac{J^2}{\sin^2\theta}} \ , \qquad (6.43)$$

which was already discussed. We just added the $J\phi_1$ term. The angle variable can be obtained now as

$$\bar\phi_1 = \frac{\partial S}{\partial J} = \phi_1 + \bar\theta - \arctan\left(\frac{J}{L}\tan\bar\theta\right) \ , \qquad (6.44)$$

where we used that L is also a function of J through $L = I_\theta + J$. This gives rise to the term $\bar\theta = \frac{\partial S}{\partial L}$. Inverting this we get the expression for ϕ_1 in terms of action–angle variables:

$$\phi_1 = \bar\varphi + \arctan\left(\frac{J}{L}\tan\bar\theta\right) \ , \qquad (6.45)$$

where we defined $\bar\varphi = \bar\phi_1 - \bar\theta$ which is the variable conjugate to J at constant L. Now we can very easily compute the time dependence of ϕ_1 at first order:

$$\phi_1 = \omega_{\bar\varphi}t + \arctan\left(\frac{J}{L}\tan\omega t\right) + [W_1, \bar\varphi + \arctan\left(\frac{J}{L}\tan\bar\theta\right)] \ , \qquad (6.46)$$

where we introduced the frequencies $\omega_{\bar\varphi} = \left(\frac{\partial E}{\partial J}\right)_L$ and $\omega = \frac{\partial E}{\partial L}$ determining the time evolution of the angle variables $\bar\varphi = \omega_{\bar\varphi}t$ and $\bar\theta = \omega t$. Evaluating the Poisson bracket in the last term we obtain the final result:

$$\phi_1 \simeq \omega_{\bar\varphi}t + \arctan\left(\frac{J}{L}\tan\omega t\right) + \frac{\lambda m^2}{4L^2}\frac{J}{L}\left(1-\frac{J^2}{L^2}\right)\frac{\sin 2\omega t}{1-\left(1-\frac{J^2}{L^2}\right)\sin^2\omega t} \ . \qquad (6.47)$$

It is instructive and useful for our purpose to put together eqs.(6.42) and (6.47) by using the parameterization

$$
\begin{aligned}
X_1 &= \sin\theta\cos\phi_1, \quad X_2 = \sin\theta\sin\phi_1 \ , \\
X_3 &= \cos\theta\cos\phi_2, \quad X_4 = \cos\theta\sin\phi_2 \ .
\end{aligned}
\tag{6.48}
$$

To obtain the time dependence of these coordinates we first use that

$$
\cos\left[\arctan\left(\frac{J}{L}\tan\omega t\right)\right] = \frac{\cos\omega t}{\sqrt{1-\left(1-\frac{J^2}{L^2}\right)\sin^2\omega t}} \ , \tag{6.49}
$$

$$
\sin\left[\arctan\left(\frac{J}{L}\tan\omega t\right)\right] = \frac{J}{L}\frac{\sin\omega t}{\sqrt{1-\left(1-\frac{J^2}{L^2}\right)\sin^2\omega t}} \ , \tag{6.50}
$$

to expand $\cos\phi_1$ and $\sin\phi_1$. Also, from (6.42) we get that

$$
\sin\theta \simeq \sqrt{1-\left(1-\frac{J^2}{L^2}\right)\sin^2\omega t} + \mathcal{O}\left(m^2\right) \ . \tag{6.51}
$$

As a result,

$$
X_1 = \cos\omega_{\bar\varphi}t \ \cos\omega t - \frac{J}{L}\sin\omega_{\bar\varphi}t \ \sin\omega t \ , \tag{6.52}
$$

$$
X_2 = \sin\omega_{\bar\varphi}t \ \cos\omega t + \frac{J}{L}\cos\omega_{\bar\varphi}t \ \sin\omega t \ , \tag{6.53}
$$

$$
X_3 = \sqrt{1-\frac{J^2}{L^2}}\sin\omega t \ \cos m\sigma \ , \tag{6.54}
$$

$$
X_4 = \sqrt{1-\frac{J^2}{L^2}}\sin\omega t \ \sin m\sigma \ , \tag{6.55}
$$

where we used that $\phi_2 = m\sigma$ and dropped all terms of order m^2 except in the frequencies ω and $\omega_{\bar\varphi}$, i.e. in the secular terms. These contribution should be described by the averaged Hamiltonian. To compare with the results of section 5 let us introduce the notation $\sin\gamma = \frac{J}{L}$ and then use the parameterization of eq.(3.3), namely

$$
X_m = a_m\cos\omega t + b_m\sin\omega t \ , \tag{6.56}
$$

where we put $\alpha = \omega t$. We can then identify

$$
\begin{aligned}
\vec{a} &= \left(\cos\omega_{\bar\varphi}t, \ \sin\omega_{\bar\varphi}t, \ 0, \ 0\right) \ , \\
\vec{b} &= \left(-\sin\gamma\sin\omega_{\bar\varphi}t, \ \sin\gamma\cos\omega_{\bar\varphi}t, \ \cos\gamma\cos m\sigma, \ \cos\gamma\sin m\sigma\right) \ ,
\end{aligned}
\tag{6.57}
$$

and thus get the following expression for V defined as in (3.4), i.e. $V = \frac{1}{\sqrt{2}}(\vec{a} - i\vec{b})$:

$$V = -\frac{i}{\sqrt{2}}(-\sin\gamma\sin\omega_{\bar\varphi}t + i\cos\omega_{\bar\varphi}t,\ \sin\gamma\cos\omega_{\bar\varphi}t + i\sin\omega_{\bar\varphi}t,\ \cos\gamma\cos m\sigma,\ \cos\gamma\sin m\sigma)$$

This is in perfect agreement with (5.24) which was obtained by solving the equations of the reduced sigma model. The only difference is an irrelevant overall factor i. The frequencies also agree since from eq.(6.22) we get

$$\omega_{\bar\varphi} = \left.\frac{\partial E}{\partial J}\right|_L \simeq -\frac{\lambda m^2}{2L^2}\frac{J}{L}\ . \tag{6.58}$$

To compare with ω in (5.23) we should note that because of the rescaling of time by $\tilde\lambda$ we need to rescale the frequencies or $m \to \frac{\sqrt{\lambda}}{L}m$. Also, we should use that $\sin\gamma = \frac{J}{L}$.

7. Conclusions

In this paper we studied strings that move fast, close to the speed of light, in the S^5 part of $AdS_5 \times S^5$ and compared them to semiclassical states of the $SO(6)$ spin chain Hamiltonian. The latter is equivalent [9] to the 1-loop, large-N, dilatation operator acting on the scalar sector of $\mathcal{N} = 4$ SYM. We showed how the classical string action can be systematically expanded in this limit, first for rotating strings and then in general. The leading order result agrees with the sigma model that we derived from the field theory by restricting to operators satisfying a "locally BPS" condition which leads to a Grassmanian $G_{2,6}$ sigma model [21, 27].

As a result, we get a precise mapping between operators and strings [4, 16, 18, 27]. In the present case, the operators are products of a large number of scalar fields. The map is such that to each portion of the string one associates a particular linear combination of the scalar fields in the product. This is done by identifying σ, the coordinate along the string, with the discrete coordinate which labels the scalar fields. In the limit (4.6) the trajectory of each point of the string is approximately a maximum circle characterized by an $SO(6)$ angular momentum which agrees with the $SO(6)$ R-charge carried by the associated scalar field. Furthermore, the maximum circle along which each point of the string is moving changes in time, and correspondingly the scalar operator also changes. This motion is the one described by the sigma model derived from the spin chain Hamiltonian or from the expansion of the string action.

Our result generalizes similar comparisons done for rotating [11, 16, 18] and pulsating [24, 15] strings, not only because it provides a map between the dual configurations, but also because it includes many more examples, for instance, fluctuations around the known solutions.

It seems feasible that we can extend this comparison to two loops. For the $SU(3)$ sector the necessary ingredients are in [30] and in section 2 of this paper. In addition, one needs to use the important observation of [31] that mixings with fermionic states can be ignored to leading order in $1/L$ expansion. For $SO(6)$, the two-loop dilatation operator needs to be computed.[26] Once the two-loop dilatation operator is known, the field theory calculation should reduce to a computation of quantum corrections in the spin chain as in [18].

On the string side the expansion to higher orders is classical and was developed here in great detail using the following method. First, one isolates a fast variable α such that the velocities of all the other coordinates can be considered small. In order to do so we have to use *phase space* description since the fast variable cannot be isolated in coordinate space. The basic reason is that the fast variable is the polar angle in the plane determined by the position *and* the momentum (see also [27] for a similar approach).

The momentum conjugate to α, which we call P_α, is large and can be used to expand the action as a power series in $1/P_\alpha$. However, since the (phase-space) Lagrangian depends on α, the momentum P_α is not conserved. To get a conserved quantity we do a series of canonical transformations that eliminate the dependence on α order by order in $1/P_\alpha$.

The new P_α is conserved and can be used to characterize the solutions. It can also be gauge fixed to a constant as can be seen more intuitively by doing a T-duality transformation along the fast variable and then fixing a static gauge. In this gauge P_α is uniformly distributed along the string and therefore can be naturally associated with the length of the spin chain. This is the parameter that is usually denoted as J for the rotating strings and L for the pulsating ones.

Expanding the final Lagrangian for large P_α is then a simple Taylor expansion. We emphasize that the difficult step is the previous one, namely, to eliminate the fast variable. This we did here only to the next to the leading order. As a technical point, we found it convenient to use the embedding Cartesian coordinates and also to introduce the fast variable α as a redundant coordinate (giving rise to a $U(1)$ gauge invariance).

The whole procedure relies on using a gauge and a set of variables which are well adapted to the expansion at the classical level. It would be interesting to study if similar simplifications occur in quantum string calculations like the ones in [44, 45].

As a check, we studied several particular examples, including pulsating and rotating solutions and were able to verify the agreement with previously known results [11, 16, 20, 21, 24, 15]. This also allowed us to obtain the coherent operators that correspond

[26] Again, the result of [31] suggest that if one is interested only in the large L limit of this dilatation operator its computation can be simplified since the operator mixing should be suppressed for $L \to \infty$.

to the pulsating solutions of [24, 15]. These are not equivalent to the eigenstates of the dilatation operator that are related to the same string solutions (have the same energy) and which were identified in [24, 15] using the Bethe ansatz. We emphasize that the operators we discussed here are not eigenstates of the dilatation operator, rather, they represent semiclassical or coherent SYM states which are naturally associated to the semiclassical string states described by the classical string solutions.

Let us mention also another result we obtained for the pulsating solutions: the exact relation between the classical energy and the conserved quantities J and L. The solution and its energy can be written in terms of elliptic integrals much in the same way as was previously done for the 2-spin rotating string solutions. Similarly, the large $\frac{L}{\sqrt{\lambda}}$ expansion becomes just a Taylor expansion and can be easily carried out. All this is achieved by identifying J and L with the action variables conjugated to the corresponding angles. This identification has also the advantage of allowing the calculation of the characteristic frequencies of motion as derivatives of the energy with respect to the action variables J and L.

Finally, from a broader perspective, the importance of understanding the duality correspondence for a large class of time-dependent semiclassical states is that it may show the way to the study of generic string oscillation modes and thus further clarify the structure of $AdS_5 \times S^5$ string spectrum. One lesson we have learned here is the crucial role of the phase space approach: the same *local* phase space effective sigma model action in which coordinates and momenta appear on an equal footing emerged from both the spin chain and the string theory. Eliminating momenta would lead to a nonlocal action.

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A. Canonical perturbation theory

In this appendix we give a brief introduction to the subject of canonical perturbation theory. The main purpose is to familiarize the reader with the notation used in the main text and to make the paper self-contained. We follow closely [37] where the interested

reader can find many more details, examples and references to the original literature. The case of non-canonical variables which is the one we studied in the main text is discussed *e.g.* in [38].

Since the equations of motion for a dynamical system cannot usually be integrated easily, perturbation theory is a widely used and well studied method. Let us consider some cases in which it can be applied. Suppose first that we have an integrable system with canonical variables $\bar{x} = (p_i, q_i)$, Hamiltonian $H(p, q)$ and Poisson brackets

$$[f, g] = \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \ . \tag{A.1}$$

We can do a canonical transformation to angle-action variables $(\bar{\theta}_i, J_i)$ such that the Hamiltonian becomes a function of the J_i only: $H(J_i)$. Hamilton's equations $\frac{d\bar{x}}{dt} = [\bar{x}, H]$ can now be easily solved:

$$J_i = \text{const} \ , \qquad \bar{\theta}_i = \omega_i t + \beta_i \ , \qquad \text{with} \ \ \omega_i = \frac{\partial H}{\partial J_i} \ . \tag{A.2}$$

Finding such a transformation, however, is not simple. If, instead, we are able to find a transformation $(p_i, q_i) \to (J_i, \theta_i)$ that puts the Hamiltonian in the form

$$H = H_0(J_i) + \epsilon H_1(J_i, \theta_i) + \epsilon^2 H_2(J_i, \theta_i) + \dots \tag{A.3}$$

with $\epsilon \ll 1$, then we can use perturbation theory in ϵ to find a further transformation $(J_i, \theta_i) \to (\bar{J}_i, \bar{\theta}_i)$ such that the Hamiltonian becomes a function of the $\bar{J}_i$ only. The ultimate objectives are: (i) to determine the characteristic frequencies $\omega_i = \partial H(\bar{J}_i)/\partial \bar{J}_i$, and (ii) to use the canonical transformation $p(\bar{\theta}_i, \bar{J}_i), q(\bar{\theta}_i, \bar{J}_i)$ to determine how these frequencies appear in the expressions for the original coordinates. All this can be done order by order in ϵ.

Below we will show how this can be done for a one-dimensional system which is always integrable. Another related example is when the motion is such that one variable changes much faster than the others. By considering the other variables "frozen" we can reduce the system to a one-dimensional one which is integrable and can be again transformed to action–angle variables perturbatively. This case is similar to the previous one only that now we should consider the motion of the other variables as perturbations. Notice that in this case it is not necessary, and can actually be inconvenient, to use action–angle variables for the "slow" coordinates.

Let us also point out that while most systems are not integrable, perturbation theory can sometimes also be used to approximate them by an integrable one. Although, generally speaking, this is the most interesting case, it will not concern us here.

Before going into further details let us discuss which is the first problem that one faces when doing perturbation theory. The problem is due to the so called secular terms and is solved by the canonical method.[27] We can illustrate it with a simple example. Consider the motion which is described by the function

$$y(t) = a \sin \omega t \ , \tag{A.4}$$

but where we know a and ω only as expansions $a = a_0 + \epsilon a_1 + \ldots$, $\omega = \omega_0 + \epsilon\omega_1 + \ldots$. At leading order the best we can do is to approximate

$$y(t) \simeq a_0 \sin(\omega_0 t + \epsilon\omega_1 t) + \epsilon a_1 \sin(\omega_0 t + \epsilon\omega_1 t) \ . \tag{A.5}$$

It would be incorrect to further expand in ϵ to get

$$y(t) \simeq a_0 \sin \omega_0 t + \epsilon a_0 \omega_1 t \cos \omega_0 t + \epsilon a_1 \sin(\omega_0 t) \ . \tag{A.6}$$

This is clear since for large t we cannot guarantee that $\epsilon\omega_1 t$ would be small. The result of the incorrect procedure is a perturbation linearly growing in t which is usually called a secular term, using nomenclature from celestial mechanics. This obvious point can become slightly more confusing if we now suppose that $y(t)$ is obtained by solving an equation of motion. The zero order solution would be $y_0(t) = a_0 \sin \omega_0 t$ and to first order we would be tempted to use the ansatz

$$y(t) \simeq a_0 \sin \omega_0 t + \epsilon y_1(t) \ . \tag{A.7}$$

Solving for $y_1(t)$ we are bound to obtain (A.6), namely the incorrect result. The natural solution is to use the ansatz where ω_0 is replaced by $\omega_0 + \epsilon\omega_1$ with ω_1 chosen so as to cancel the term linear in t. The physical interpretation of this is clear. If we think of two particles, one following an unperturbed trajectory and the other the perturbed one, even though the trajectories are close, one particle lags behind the other because of the frequency correction, producing the linear term in the difference of positions.

Canonical perturbation theory deals with this problem in the following way. At each order in perturbation theory one has a Hamiltonian $H_n(\theta, J)$ which can be divided into an averaged and a fluctuating part as follows

$$\begin{aligned} H_n(\theta, J) &= \langle H_n \rangle(J) + \{H_n\}(\theta, J) \ , \\ \langle H_n \rangle(J) &= \frac{1}{2\pi} \int_0^{2\pi} d\theta \ H_n(\theta, J) \ , \\ \{H_n\}(\theta, J) &= H_n - \langle H_n \rangle(J) \ . \end{aligned} \tag{A.8}$$

[27]This not the only problem that perturbation theory has to face. Another well-known problem that we will not discuss here is related to the so called small denominators.

The fluctuating part can be eliminated by a canonical transformation to get a Hamiltonian depending on J only (at this order). If we try to eliminate also $\langle H_n \rangle(J)$ this gives rise to secular terms. Instead, since $\langle H_n \rangle(J)$ depends only on J, it can be incorporated into the transformed Hamiltonian and therefore gives a correction to the frequency $\omega = \frac{\partial H}{\partial J}$.

Let us now consider this procedure in more detail. We have a system with canonical variables (θ, J) that we want to transform into $(\bar{\theta}, \bar{J})$ such that the Hamiltonian

$$H = H_0(J) + \epsilon H_1(J, \theta) + \epsilon^2 H_2(J, \theta) + \dots \tag{A.9}$$

becomes independent of $\bar{\theta}$. Denoting $\bar{x} = (J, \theta)$, such transformation $T_\epsilon : \bar{x} \to \bar{x}_\epsilon$ will be generated by a function $W(\bar{x}, \epsilon)$

$$\frac{d\bar{x}}{d\epsilon} = [\bar{x}, W(\bar{x}, \epsilon)] \ . \tag{A.10}$$

Solving for $\bar{x}(\epsilon)$ gives a canonical transformation $\bar{x}(0) \to \bar{x}(\epsilon)$. For small ϵ we can expand W as

$$W = W_1 + \epsilon W_2 + \dots \ . \tag{A.11}$$

Expanding also $\bar{x}$ in powers of ϵ and taking into account that the functions W_n should be further expanded since they depend on $\bar{x}$, we can solve (A.10) obtaining

$$\bar{x}(\epsilon) = \bar{x} - \epsilon[W_1, \bar{x}] - \frac{1}{2}\epsilon^2 [W_2, \bar{x}] + \frac{1}{2}\epsilon^2 [W_1, [W_1, \bar{x}]] + \mathcal{O}(\epsilon^3) \ , \tag{A.12}$$

where on the right hand side all functions are evaluated at $\epsilon = 0$. An arbitrary function $F(\bar{x})$ transforms in such a way that $F_\epsilon(\bar{x}_\epsilon) = F(\bar{x})$. This condition and (A.12) imply that

$$F_\epsilon = F + \epsilon[W_1, F] + \frac{1}{2}\epsilon^2 [W_2, F] + \frac{1}{2}\epsilon^2 [W_1, [W_1, F]] + \mathcal{O}(\epsilon^3) \ . \tag{A.13}$$

If we now put $F = H$ with H being the Hamiltonian of eq.(A.9) we get the transformed Hamiltonian

$$H_\epsilon = H_0 + \epsilon H_1 + \epsilon^2 H_2 + \epsilon[W_1, H_0] + \epsilon^2 [W_1, H_1] + \frac{1}{2}\epsilon^2 [W_2, H_0] + \frac{1}{2}\epsilon^2 [W_1 [W_1, H_0]] + \mathcal{O}(\epsilon^3) \ . \tag{A.14}$$

The left hand side should be independent of θ so we should choose W_1 and W_2 to cancel the θ dependence on the right hand side. As discussed before, we separate each term on the right hand side into averaged and fluctuating parts using (A.9) in order to avoid secular terms. This leads to the equations

$$[W_1, H_0] = \frac{\partial H_0}{\partial J} \frac{\partial W_1}{\partial \theta} = -\{H_1\}, \tag{A.15}$$

$$[W_2, H_0] = \frac{\partial H_0}{\partial J} \frac{\partial W_2}{\partial \theta} = -2\{H_2\} - 2[W_1, \langle H_1 \rangle] - [W_1, \{H_1\}], \tag{A.16}$$

which determine W_1 and W_2. Notice that attempting to eliminate the average value $\langle H_1 \rangle$ give rise to a term linear in θ in W_1, namely a secular term (since $\theta = \omega t$). The transformed Hamiltonian is given by the averaged terms that remain after the transformation

$$\bar{H} = H_0 + \epsilon \langle H_1 \rangle + \epsilon^2 \langle H_2 \rangle + \frac{1}{2} \langle [W_1, \{H_1\}] \rangle \ , \tag{A.17}$$

and it determines the frequencies. Inverting the canonical transformation (A.12) allows one to immediately write down the time dependence of the original variables.

As a final comment, let us point out that although, at first sight, this procedure may seem involved, it easy to convince oneself by working out an example that it is much easier than a more straightforward approach. One of the main points is that the coordinate transformations used are canonical and therefore are defined by just one scalar function. Trying to solve the equations directly requires obtaining a separate function for each of the coordinates. This soon becomes intractable. A very simple example is actually the one we worked out in section 6.4 and it can be used to illustrate all the points discussed here.

References

[1] J. Maldacena, "The large N limit of superconformal field theories and supergravity," Adv. Theor. Math. Phys. **2**, 231 (1998) [Int. J. Theor. Phys. **38**, 1113 (1998)], `hep-th/9711200`,
S. S. Gubser, I. R. Klebanov and A. M. Polyakov, "Gauge theory correlators from non-critical string theory," Phys. Lett. B **428**, 105 (1998), [hep-th/9802109],
E. Witten, "Anti-de Sitter space and holography," Adv. Theor. Math. Phys. **2**, 253 (1998), [hep-th/9802150],
O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, "Large N field theories, string theory and gravity," Phys. Rept. **323**, 183 (2000)

[2] G. 't Hooft, "A Planar Diagram Theory For Strong Interactions," Nucl. Phys. B **72**, 461 (1974), "A Two-Dimensional Model For Mesons," Nucl. Phys. B **75**, 461 (1974).

[3] A. M. Polyakov, "Gauge fields and space-time," Int. J. Mod. Phys. A **17S1**, 119 (2002) [hep-th/0110196].

[4] D. Berenstein, J. M. Maldacena and H. Nastase, "Strings in flat space and pp waves from N =4 super Yang Mills," JHEP **0204**, 013 (2002) [hep-th/0202021].

[5] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, "A semi-classical limit of the gauge/string correspondence," Nucl. Phys. B **636**, 99 (2002) [hep-th/0204051].

[6] S. Frolov and A. A. Tseytlin, "Semiclassical quantization of rotating superstring in $AdS_5 \times S^5$," JHEP **0206**, 007 (2002) [hep-th/0204226].

[7] S. Frolov and A. A. Tseytlin, "Multi-spin string solutions in $AdS_5 \times S^5$," Nucl. Phys. B **668**, 77 (2003) [hep-th/0304255].

[8] A. A. Tseytlin, "Spinning strings and AdS/CFT duality," hep-th/0311139.

[9] J. A. Minahan and K. Zarembo, "The Bethe-ansatz for N = 4 super Yang-Mills," JHEP **0303**, 013 (2003) [hep-th/0212208].

[10] N. Beisert, J. A. Minahan, M. Staudacher and K. Zarembo, "Stringing spins and spinning strings," JHEP **0309**, 010 (2003) [hep-th/0306139].

[11] N. Beisert, S. Frolov, M. Staudacher and A. A. Tseytlin, "Precision spectroscopy of AdS/CFT," JHEP **0310**, 037 (2003) [hep-th/0308117].

[12] G. Arutyunov, S. Frolov, J. Russo and A. A. Tseytlin, "Spinning strings in $AdS_5 \times S^5$ and integrable systems," Nucl. Phys. B **671**, 3 (2003) [hep-th/0307191].

[13] G. Arutyunov, J. Russo and A. A. Tseytlin, "Spinning strings in $AdS_5 \times S^5$: New integrable system relations," Phys. Rev. D **69**, 086009 (2004) [hep-th/0311004].

[14] G. Arutyunov and M. Staudacher, "Matching higher conserved charges for strings and spins," JHEP **0403**, 004 (2004) [hep-th/0310182]. "Two-loop commuting charges and the string / gauge duality," hep-th/0403077.

[15] J. Engquist, J. A. Minahan and K. Zarembo, "Yang-Mills duals for semiclassical strings on $AdS_5 \times S^5$," JHEP **0311**, 063 (2003) [hep-th/0310188].

[16] M. Kruczenski, "Spin chains and string theory," hep-th/0311203.

[17] V. A. Kazakov, A. Marshakov, J. A. Minahan and K. Zarembo, "Classical/quantum integrability in AdS/CFT," hep-th/0402207.

[18] M. Kruczenski, A. V. Ryzhov and A. A. Tseytlin, "Large spin limit of $AdS_5 \times S^5$ string theory and low energy expansion of ferromagnetic spin chains," hep-th/0403120.

[19] D. Serban and M. Staudacher, "Planar N = 4 gauge theory and the Inozemtsev long range spin chain," JHEP **0406**, 001 (2004) [hep-th/0401057].

[20] R. Hernandez and E. Lopez, "The SU(3) spin chain sigma model and string theory," JHEP **0404**, 052 (2004) [hep-th/0403139].

[21] B. J. Stefanski, Jr. and A. A. Tseytlin, "Large spin limits of AdS/CFT and generalized Landau-Lifshitz equations," JHEP **0405**, 042 (2004) [hep-th/0404133].

[22] N. Beisert and M. Staudacher, “The N=4 SYM integrable super spin chain”, Nucl. Phys. B **670**, 439 (2003) [hep-th/0307042].

[23] C. Kristjansen, “Three-spin strings on AdS(5) x S**5 from N = 4 SYM,” Phys. Lett. B **586**, 106 (2004) [hep-th/0402033]. L. Freyhult, “Bethe ansatz and fluctuations in SU(3) Yang-Mills operators,” hep-th/0405167. C. Kristjansen and T. Mansson, “The Circular, Elliptic Three Spin String from the SU(3) Spin Chain,” hep-th/0406176.

[24] J. A. Minahan, “Circular semiclassical string solutions on $AdS_5 \times S^5$,” Nucl. Phys. B **648**, 203 (2003) [hep-th/0209047].

[25] D. Mateos, T. Mateos and P. K. Townsend, “Supersymmetry of tensionless rotating strings in $AdS_5 \times S^5$, and nearly-BPS operators,” hep-th/0309114. “More on supersymmetric tensionless rotating strings in $AdS_5 \times S^5$,” hep-th/0401058.

[26] A. Mikhailov, “Speeding strings,” JHEP **0312**, 058 (2003) [hep-th/0311019].

[27] A. Mikhailov, “Slow evolution of nearly-degenerate extremal surfaces,” hep-th/0402067.

[28] A. Mikhailov, “Supersymmetric null-surfaces,” hep-th/0404173.

[29] E. H. Fradkin, “Field Theories Of Condensed Matter Systems,” Redwood City, USA: Addison-Wesley (1991) 350 p. (Frontiers in physics, 82). I. Affleck, “Quantum spin chains and the Haldane gap”, J. Phys C 1 (1989), 3047

[30] N. Beisert, “The su(2|3) dynamic spin chain,” hep-th/0310252.

[31] J. A. Minahan, “Higher loops beyond the SU(2) sector,” hep-th/0405243.

[32] S. Frolov and A. A. Tseytlin, “Rotating string solutions: AdS/CFT duality in non-supersymmetric sectors,” Phys. Lett. B **570**, 96 (2003) [hep-th/0306143].

[33] H. J. de Vega, A. L. Larsen and N. Sanchez, “Semiclassical quantization of circular strings in de Sitter and anti-de Sitter space-times,” Phys. Rev. D **51**, 6917 (1995) [hep-th/9410219].

[34] H. Dimov and R. C. Rashkov, “Generalized pulsating strings,” hep-th/0404012.

[35] U. H. Danielsson, A. Guijosa and M. Kruczenski, “IIA/B, wound and wrapped,” JHEP **0010**, 020 (2000), [hep-th/0009182],
J. Gomis and H. Ooguri, “Non-relativistic closed string theory,” J. Math. Phys. **42**, 3127 (2001), [hep-th/0009181].

[36] H. Eichenherr and M. Forger, “On The Dual Symmetry Of The Nonlinear Sigma Models,” Nucl. Phys. B **155**, 381 (1979). “Higher Local Conservation Laws For Nonlinear Sigma Models On Symmetric Spaces,” Commun. Math. Phys. **82**, 227 (1981).

[37] A.J. Lichtenberg, M.A. Lieberman, "Regular and chaotic dynamics", Second Edition, Springer-Verlag (1992), New York, NY, USA.

[38] R.G. Littlejohn, "A guiding Center Hamiltonian: A new approach" J. Math. Phys. **20**, 2445 (1979).

[39] N. Beisert, C. Kristjansen and M. Staudacher, "The dilatation operator of N = 4 super Yang-Mills theory," Nucl. Phys. B **664**, 131 (2003) [hep-th/0303060].

[40] W. M. Zhang, D. H. Feng and R. Gilmore, "Coherent States: Theory And Some Applications," Rev. Mod. Phys. **62**, 867 (1990).

[41] A. Jevicki and N. Papanicolaou, "Semiclassical Spectrum Of The Continuous Heisenberg Spin Chain," Annals Phys. **120**, 107 (1979).

[42] M. Smedback, "Pulsating strings on $AdS_5 \times S^5$," hep-th/0405102.

[43] A. Khan and A. L. Larsen, "Spinning pulsating string solitons in $AdS_5 \times S^5$," Phys. Rev. D **69**, 026001 (2004) [hep-th/0310019].

[44] A. A. Tseytlin, "Semiclassical quantization of superstrings: $AdS_5 \times S^5$ and beyond," Int. J. Mod. Phys. A **18**, 981 (2003) [hep-th/0209116].

[45] S. Frolov and A. A. Tseytlin, "Quantizing three-spin string solution in $AdS_5 \times S^5$," JHEP **0307**, 016 (2003) [hep-th/0306130].

[46] A. Gorsky, "Spin chains and gauge / string duality," hep-th/0308182.

[47] I.S. Gradshteyn, I.M. Ryzhik, "Table of Integrals Series and Products", Sixth edition, Academic Press (2000), San Diego, CA, USA, London, UK.